

Quantum corrections to short folded superstring in $AdS_3 \times S^3 \times M^4$

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Abstract

We consider integrable superstring theory on $AdS_3 \times S^3 \times M^4$ where $M^4 = T^4$ or $M^4 = S^3 \times S^1$ with generic ratio of the radii of the two 3-spheres. We compute the one-loop energy of a short folded string spinning in AdS_3 and rotating in S^3 . The computation is performed by world-sheet small spin perturbation theory as well as by quantizing the classical algebraic curve characterizing the finite-gap equations. The two methods give equal results up to regularization contributions that are under control. One important byproduct of the calculation is the part of the energy which is due to the dressing phase in the Bethe Ansatz. Remarkably, this contribution E_1^{dressing} turns out to be independent on the radii ratio. In the $M^4 = T^4$ limit, we discuss how E_1^{dressing} relates to a recent proposal for the dressing phase tested in the $\mathfrak{su}(2)$ sector. We point out some difficulties suggesting that quantization of the AdS_3 classical finite-gap equations could be subtler than the easier $AdS_5 \times S^5$ case.

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1 Introduction and summary

The Maldacena correspondence between quantum strings in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric gauge theory has been explored in recent years by means of the powerful unifying framework of integrability [1]. Integrable structures can be formulated in a non-perturbative way and allow to analyze the weak-strong coupling connection in great details. The technical machinery of integrability¹ is a promising tool to study similar less supersymmetric cases of the duality like superstring in $AdS_3 \times S^3 \times M^4$ and $AdS_2 \times S^2 \times M^6$ supported by R-R fluxes. For these gravitational backgrounds the dual superconformal theories are poorly understood [2, 3] and it is less straightforward to identify the underlying non-perturbative integrable structures, in particular the Bethe Ansatz equations.

Recent important progress has been done in the case of strings on $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$. They are described by the GS superstring action on the supercosets $PSU(1, 1|2) \times PSU(1, 1|2)/SU(1, 1) \times SU(2)$ and $D(2, 1; \alpha)^2/SU(1, 1) \times SU(2)^2$. The first model may be viewed as a special case of the second. If the radius of AdS_3 is set to 1, then the radii of the two 3-spheres can be parametrized as $R_1^2 = \alpha^{-1}$, $R_2^2 = (1 - \alpha)^{-1}$, i.e. the $AdS_3 \times S^3 \times T^4$ model with $R_2 = \infty$ corresponds to $\alpha = 1$.

In [4], a set of asymptotic Bethe Ansatz (ABA) equations was proposed for these models starting from the classical integrable supercoset sigma model and conjecturing a natural discretisation of the corresponding finite-gap equations following closely the analogy with the $AdS_5 \times S^5$ case [5] (see [6])². The ABA contains an undetermined dressing phase possibly equal to the BES phase [10] appearing in the $AdS_5 \times S^5$ case (as well as in $AdS_4 \times \mathbb{CP}^3$ [11]).

The ABA system has been analyzed more deeply in [12, 13, 8, 14]. In particular, it was

¹This includes for instance the algebraic curve description of the string classical solutions, the excitation S -matrix, the asymptotic Bethe Ansatz equations and their TBA extensions for the calculation of finite size effects.

²We remark that the spectrum described by the finite gap equations is missing two massless modes. One mode corresponds to excitations on S^1 , and the other to a mode shared by the two spheres that is not present in the coset model since the Virasoro constraints are overimposed there. These modes can be put back by hand at the classical level, but it is not yet clear how to do that at the quantum level [7, 8]. In the example discussed in this paper, massless modes cancel out in a conventional world-sheet computation suggesting that the supercoset description is indeed a consistent truncation like it happens for instance for the Bethe equations for the $\mathfrak{su}(2)$ sector of AdS_5/CFT_4 that can be reconstructed from the finite-gap equations on $S^3 \times \mathbb{R}^1$ [9], which ignores most part of the string modes in $AdS_5 \times S^5$.

claimed in [14] that here one cannot fix the dressing scalar factors in the magnon S-matrix using crossing symmetry as was done in the $AdS_5 \times S^5$ case [15, 16, 17]. This statement does not rule out the simplest scenario where the phase is given by the BES expression. However, very recent further developments in [7, 18] made the story more involved by concluding that there should be several scalar phase factors and that they may differ from the BES expression. Similar conclusions appeared in [19] and point out the necessity of a new phase.

A first study of the specific form of the AdS_3 phase appeared in [20] where a proposal (BLMT) has been suggested for the leading quantum string correction to the classical AFS phase in the ABA system of [4, 13]. The BLMT phase was derived by mimicking the analogous steps originally applied to the $AdS_5 \times S^5$ case [21, 22, 23, 24] and is based on the study of the phase-dependent ABA predictions to the quantum string and algebraic curve computations of the 1-loop corrections to semiclassical string energies³. In particular, the simple example of a rigid circular string in $AdS_3 \times S^3 \subset AdS_3 \times S^3 \times T^4$ with two equal spins in S^3 [29] has been discussed in [20], together with the closely related (via analytic continuation) case of (S, J) folded long string [30]. The conclusion of [20] was a proposal for the phase in the ABA of [4, 13] that is related but differs from the standard BES form of [21, 22].

Unfortunately, the analysis of [20] has some important loose ends. Indeed, the derivation of the phase requires some *ad hoc* steps that work well for the $SU(2)$ circular string, but have an unclear meaning for more general solutions. Quantization of any classical string solution amounts to finding the frequencies of the eigenmodes of the classical equations of motion, promote them to quantum oscillators with definite frequencies and sum over the zero point energies $\frac{1}{2} \sum (-1)^F \omega$. Frequencies can be found by a standard world-sheet analysis or by perturbing the classical algebraic curve. The same set of frequencies should be found, up to trivial changes canceling in the sum. Nevertheless, an infinite summation is involved since each eigenmode has an associated discrete momentum. Different prescriptions for the sum can be used. The most natural in the world-sheet approach is the one that knows nothing about integrability and simply sets a common cut-off on the momentum of each eigenmode. On the other hand, the natural prescription when quantizing the algebraic curve is different and assigns a common cut-off on the spectral radius variable associated with each mode. The two prescriptions lead to a finite difference. We shall refer to this effect naming it a *regularization mismatch*. This ambiguity should be fixed, in principle and as usual, by fixing finite renormalizations in order to implement the symmetries of the problem. In the problem at hand, symmetries are closely related to the integrable structure and one is led to the hope that properly fixing the latter will accomplish the desired finite renormalization.

Indeed, this is what happens in the $SU(2)$ circular string [20]. Quantization of the algebraic curve leads to a dressing effect that can be written as an additional one-loop piece \mathcal{V} in the finite-gap equations precisely as in the $AdS_5 \times S^5$ case [24]. However, in AdS_3 , the new piece apparently cannot be immediately interpreted as a phase correction in the Bethe equations. Instead $\mathcal{V} = \mathcal{V}_{\text{phase}} + \delta\mathcal{V}$, where only $\mathcal{V}_{\text{phase}}$ admits such an interpretation. Remarkably, for the $SU(2)$ circular string, the extra term $\delta\mathcal{V}$ happens to cancel exactly the regularization mismatch for yet unclear reasons. The conclusion is that, for this particular solution, we can enforce dressing effects to be solely encoded in a phase in the quantum Bethe equations. This fixes the

³Previous semiclassical computations for superstrings in $AdS_3 \times S^3 \times M^4$ can be found in [25, 26, 27, 28].

regularization ambiguity and confirms the world-sheet dressing energy.

In principle, this could be an accident. The discretization/quantization of the unambiguous finite-gap classical string Bethe equations is non trivial. The construction of the discrete all-loop Bethe Ansatz should match the two sides of the AdS/CFT correspondence and there could be space for non-trivial new features compared to the AdS_5 case. Difficulties with the naive quantization are indeed clearly discussed in [18] by comparing the semiclassical finite-gap equations with the near-BMN spectrum.

The aim of this paper is that of making new steps toward a clarification of this issue by considering a different classical solution of string theory on $AdS_3 \times S^3 \times S^1 \times S^1$ and studying its one-loop energy with particular attention to the contributions related to the dressing phase. We focus on a folded string spinning in AdS_3 with semiclassical spin \mathcal{S} and rotating in S^3 with angular momentum \mathcal{J} . The short string regime (small \mathcal{S} at generic \mathcal{J}) is particularly interesting and much experience is known in the analogous $AdS_5 \times S^5$ case [31, 32, 33] or $AdS_4 \times \mathbb{CP}^3$ [34]. In particular, each term of the small \mathcal{S} expansion of the energy can be considered for large angular momentum \mathcal{J} . In this limit, the dressing terms are neatly separated out and very interesting connections can be studied with the weak-coupling Bethe equations as discussed in [35].

We study the one-loop energy for a generic α -dependent geometry and find that the dressing energy is independent on α after the very same redefinition of string tension found in [20, 19] for other classical solution. This redefinition is interpreted as a simple rewriting of the energy in terms of the interpolating coupling appearing in the Bethe Ansatz equations. The calculation is done both with world-sheet methods and with quantization of the algebraic curve with perfect agreement. Going to the T^4 limit, where we have the BLMT proposal for the phase, we investigate the interplay of the extra term $\delta\mathcal{V}$ and the regularization mismatch finding that they do not balance in this case. This means that we cannot fix the regularization ambiguity in a satisfactory way. The two different $M^4 = T^4$ results described above are ⁴

$$\begin{aligned} \text{WS} \equiv \text{AC-reg. mismatch : } \quad E_1^{\text{dressing}} &= \frac{\coth^{-1}(\sqrt{\mathcal{J}^2 + 1})}{2\mathcal{J}^3\sqrt{\mathcal{J}^2 + 1}} \mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3), \\ \text{BA with BLMT phase : } \quad E_1^{\text{dressing}} &= \left[\frac{\coth^{-1}(\sqrt{\mathcal{J}^2 + 1})}{2\mathcal{J}^3\sqrt{\mathcal{J}^2 + 1}} + \frac{1}{2\mathcal{J}^4\sqrt{\mathcal{J}^2 + 1}} \right] \mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3). \end{aligned}$$

Thus, a deeper and more complete understanding of the role of $\delta\mathcal{V}$ is necessary in the general case. This statement can be rephrased by saying that at the moment there is no clear matching between the quantization of the finite-gap equations and the explicit string spectrum. It would be very interesting to test a different set of ABA equations conjectured for the $M^4 = T^4$ case like those derived in [7, 18] for $0 < \alpha < 1$.

The outline of the paper is the following. In Sec. (2) we derive the one-loop energy for the folded string in α -dependent $AdS_3 \times S^3 \times S^1 \times S^1$ by world-sheet perturbation theory. In Sec. (3), we derive the same set of frequencies by quantizing the classical algebraic curve and discuss the relation between the two approaches by computing the regularization mismatch. In Sec. (4),

⁴WS = world-sheet computation with common cut-off on the momenta of all modes, AC = algebraic curve quantization with common spectral radius cut-off.

we match the large \mathcal{J} expansion of the string dressing energy from the weak-coupling large $J = \sqrt{\lambda} \mathcal{J}$ expansion of the one and two-loop. Finally, in Sec. (5), we explore the possibility of recovering the dressing one-loop energy from a suitable dressing phase in the Bethe equations. Various appendices are devoted to technical details.

2 Folded string in $AdS_3 \times S^3 \times S^1 \times S^1$

2.1 Classical solution

The classical solution we are going to consider is described in Appendix C of [26] and has the same form as the analogous folded string solution in $AdS_5 \times S^5$ [30]. The metric of $AdS_3 \times S^3 \times S^1 \times S^1$ is parametrized as

$$ds^2 = R^2 \left(ds_{\text{AdS}}^2 + \frac{1}{\alpha} ds_{S_+^3}^2 + \frac{1}{1-\alpha} ds_{S_-^3}^2 \right) + dU^2, \quad (2.1.1)$$

$$ds_{\text{AdS}}^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2, \quad (2.1.2)$$

$$ds_{S_\pm^3}^2 = d\beta_\pm^2 + \cos^2 \beta_\pm (d\gamma_\pm^2 + \cos^2 \gamma_\pm d\varphi_\pm^2). \quad (2.1.3)$$

The relation between the Anti de Sitter radius R and the radii of the two S^3 spheres is a consequence of the supergravity equations of motion.

The classical folded string solution follows from the Ansatz

$$t = \kappa \tau, \quad \phi = w \tau, \quad \rho(\sigma) = \rho(\sigma + 2\pi), \quad (2.1.4)$$

$$\varphi_\pm = \mathcal{J}_\pm \tau, \quad \gamma_\pm = \beta_\pm = U = 0. \quad (2.1.5)$$

We shall assume the following distribution of the sphere angular momenta \mathcal{J}_\pm between the two spheres

$$\mathcal{J}_+ = \alpha \mathcal{J}, \quad \mathcal{J}_- = (1 - \alpha) \mathcal{J}. \quad (2.1.6)$$

This choice amounts to have a well-defined BPS limit for $\mathcal{S} = 0$. The equation of motion for $\rho(\sigma)$ and the Virasoro constraint takes the same form as in $AdS_5 \times S^5$. In particular, we have

$$\rho'^2 = \kappa^2 \cosh^2 \rho - w^2 \sinh^2 \rho - \mathcal{J}^2. \quad (2.1.7)$$

The coordinate ρ varies from 0 to its maximal value ρ_*

$$\coth^2 \rho_* = \frac{w^2 - \mathcal{J}^2}{\kappa^2 - \mathcal{J}^2} \equiv 1 + \frac{1}{\varepsilon^2}, \quad (2.1.8)$$

where ε measures the length of the string and is small in the short string limit. The solution of the differential equation for ρ , *i.e.*

$$\rho' = \pm \sqrt{\kappa^2 - \mathcal{J}^2} \sqrt{1 - \varepsilon^{-2} \sinh^2 \rho}, \quad \rho(0) = 0, \quad (2.1.9)$$

can be written in terms of a Jacobi elliptic function

$$\sinh \rho = \varepsilon \operatorname{sn}(\sqrt{\kappa^2 - \mathcal{J}^2} \varepsilon^{-1} \sigma, -\varepsilon^2). \quad (2.1.10)$$

The periodicity condition and the charges are

$$\begin{aligned}\sqrt{\kappa^2 - \mathcal{J}^2} &= \varepsilon \, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -\varepsilon^2\right), & \mathcal{E}_0 &\equiv \frac{E_0}{\sqrt{\lambda}} = \frac{\kappa}{\sqrt{\kappa^2 - \mathcal{J}^2}} \varepsilon \, {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; -\varepsilon^2\right), \\ \mathcal{S} &\equiv \frac{S}{\sqrt{\lambda}} = \frac{w}{\sqrt{\kappa^2 - \mathcal{J}^2}} \frac{\varepsilon^3}{2} {}_2F_1\left(\frac{1}{2}, \frac{3}{2}; 2; -\varepsilon^2\right), & J &= \mathcal{J}\sqrt{\lambda}.\end{aligned}\tag{2.1.11}$$

2.2 One-loop correction to the energy: World-sheet computation

The one-loop energy is obtained according to the following general recipe. First, we compute the Lagrangian for the quadratic fluctuations by shifting all fields $\Phi = (t, \phi, \rho, \dots)$ with respect to their classical values and evaluating the action at quadratic level

$$\mathcal{S}[\Phi] = \mathcal{S}\left[\Phi_{\text{classical}} + \frac{1}{\sqrt{\lambda}}\tilde{\Phi}\right] = \mathcal{S}[\Phi_{\text{classical}}] + \int d\tau d\sigma \tilde{\Phi}^T \mathcal{D}(\partial_\sigma, \partial_\tau, \sigma) \tilde{\Phi} + \dots\tag{2.2.1}$$

Then, for each integer mode number n , we consider a universal time dependence $e^{i\omega_n \tau}$ and find the perturbative solution to the equation

$$\mathcal{D}(\partial_\sigma, i\omega_n, \sigma) \tilde{\Phi}_n^I(\sigma) = 0,\tag{2.2.2}$$

where

$$\tilde{\Phi}_n^I = e^{in\sigma} \tilde{\Phi}_n^{I(0)} + \varepsilon \tilde{\Phi}_n^{I(1)}(\sigma) + \varepsilon^2 \tilde{\Phi}_n^{I(2)}(\sigma) + \dots,\tag{2.2.3}$$

$$\omega_n^I = \omega_n^{I(0)} + \varepsilon \omega_n^{I(1)} + \varepsilon^2 \omega_n^{I(2)} + \dots\tag{2.2.4}$$

This expansion has a solution for certain constant vectors $\tilde{\Phi}_n^{I(0)}$ that are associated, as I varies, to the various fields of the problem. The other coefficient functions $\tilde{\Phi}_n^{I(1,2,\dots)}$ follow from the equation. Finally, we obtain the one-loop energy, but summing over the properly normalized zero point energies

$$E_1 = \frac{1}{2\kappa} \sum_I (-1)^{F_I} \sum_{n \in \mathbb{Z}} \omega_n^I \equiv \frac{1}{2\kappa} \mathfrak{S},\tag{2.2.5}$$

where $(-1)^{F_I}$ is the Bose-Fermi sign of the field. For later use, we have denoted by \mathfrak{S} the signed sum over frequencies without the factor $1/(2\kappa)$.

2.2.1 Bosonic quadratic fluctuations

The bosonic fluctuations are discussed in [26] in the static gauge. Denoting by a tilde the fluctuations, the static gauge fixes $\tilde{t} = \tilde{\rho} = 0$. Introducing the linear combinations

$$\varphi = \varphi_+ + \varphi_-, \quad \psi = -\sqrt{\frac{1-\alpha}{\alpha}} \varphi_+ + \sqrt{\frac{\alpha}{1-\alpha}} \varphi_-, \tag{2.2.6}$$

there are then two coupled fields $(\tilde{\phi}, \tilde{\varphi})$, two massless fields $\tilde{\psi}, \tilde{U}$, and the decoupled massive fields $\tilde{\beta}_{\pm}, \tilde{\gamma}_{\pm}$ with equal masses \mathcal{J}_{\pm} . The coupled sector in the static gauge is described by ⁵

$$\mathcal{L}_{\text{coupled}} = -\partial^a a_3 \partial_a a_3 - m_3^2 a_3^2 - \partial^a a_4 \partial_a a_4 - m_4^2 a_4^2 + \frac{4w\kappa\mathcal{J}}{\mathcal{J}^2 + \rho'^2} a_4 \partial_{\tau} a_3, \quad (2.2.7)$$

where

$$m_3^2 = \mathcal{J}^2 + 2\rho'^2 + \frac{2w^2\kappa^2}{\mathcal{J}^2 + \rho'^2} - \frac{3w^2\kappa^2\mathcal{J}^2}{(\mathcal{J}^2 + \rho'^2)}, \quad (2.2.8)$$

$$m_4^2 = \mathcal{J}^2 \left[-1 + \frac{2(w^2 + \kappa^2)}{\mathcal{J}^2 + \rho'^2} - \frac{3w^2\kappa^2}{(\mathcal{J}^2 + \rho'^2)^2} \right]. \quad (2.2.9)$$

Thus, $\mathcal{L}_{\text{coupled}} = (a_3, a_4) Q_B (a_3, a_4)^T$, with

$$Q_B = \begin{pmatrix} \partial^a \partial_a - m_3^2 & -\frac{2w\kappa\mathcal{J}}{\mathcal{J}^2 + \rho'^2} \partial_{\tau} \\ \frac{2w\kappa\mathcal{J}}{\mathcal{J}^2 + \rho'^2} \partial_{\tau} & \partial^a \partial_a - m_4^2 \end{pmatrix}. \quad (2.2.10)$$

2.2.2 Fermionic quadratic fluctuations

The operator that describes the quadratic fermionic fluctuations is

$$\begin{aligned} \mathcal{D}_F = & \Gamma^a \partial_a - \frac{\kappa w \mathcal{J}}{2(\rho'^2 + \mathcal{J}^2)} \Gamma^{12} \bar{\Gamma} + \frac{\rho'}{2} (\Gamma^{012} - \sqrt{\alpha} \Gamma^{345} - \sqrt{1-\alpha} \Gamma^{678}) \\ & + \frac{\sqrt{\rho'^2 + \mathcal{J}^2} - \rho'}{2} (\Gamma^{012} - (\alpha \Gamma^{34} + (1-\alpha) \Gamma^{67}) \bar{\Gamma}), \end{aligned} \quad (2.2.11)$$

where

$$\bar{\Gamma} = \sqrt{\alpha} \Gamma^5 + \sqrt{1-\alpha} \Gamma^8. \quad (2.2.12)$$

This operator is rather complicated, but can be simplified as explained in App. (A).

2.3 Short string expansion of frequencies in the flat space limit

Bosons

It is convenient to rotate the Q_B operator as

$$Q_B \rightarrow R_B^{-1} Q_B R_B, \quad R_B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{i}{2} \end{pmatrix}. \quad (2.3.1)$$

Then $Q_B = Q_B^{(0)} + \varepsilon Q_B^{(1)} + \dots$ and

$$Q_B^{(0)} = \begin{pmatrix} -n^2 + \omega^2 - 2\sqrt{\mathcal{J}^2 + 1}\omega - 1 & 0 \\ 0 & -n^2 + \omega^2 + 2\sqrt{\mathcal{J}^2 + 1}\omega - 1 \end{pmatrix}, \quad (2.3.2)$$

⁵Notice that for quadratic fluctuations the Pohlmeyer reduction [36] (absorbing a global factor 2 in the normalization of the fields) gives the same result.

when acting on functions $\sim e^{in\sigma}$. At finite $\mathcal{J} > 0$, the eigenvalues of $Q_B^{(0)}$ can be written as

$$\pm(\sqrt{n^2 + \mathcal{J}^2} + \sqrt{1 + \mathcal{J}^2}), \quad \pm(\sqrt{n^2 + \mathcal{J}^2} - \sqrt{1 + \mathcal{J}^2}). \quad (2.3.3)$$

Hence, the coupled system contributes the following set of $\omega > 0$ frequencies

$$\sqrt{n^2 + \mathcal{J}^2} \pm \sqrt{1 + \mathcal{J}^2}. \quad (2.3.4)$$

To these two frequencies, we have to add the two massless and two massive decoupled contributions that read

$$2 \times \left\{ n, \quad \sqrt{n^2 + \alpha^2 \mathcal{J}^2}, \quad \sqrt{n^2 + (1 - \alpha)^2 \mathcal{J}^2} \right\}. \quad (2.3.5)$$

Fermions

Evaluating at $\varepsilon = 0$ the frequencies coming from the 8 blocks in which we decomposed $-\mathcal{D}_F^2$ (see App. (A)), we find after some calculation the following fermionic frequencies (plus the opposite one due to the symmetry $\omega \rightarrow -\omega$)

$$n \pm \frac{1}{2} \sqrt{\mathcal{J}^2 + 1}, \quad (2.3.6)$$

$$\sqrt{n^2 + \mathcal{J}^2} \pm \frac{1}{2} \sqrt{\mathcal{J}^2 + 1}, \quad (2.3.7)$$

$$\sqrt{n^2 + \alpha^2 \mathcal{J}^2} \pm \frac{1}{2} \sqrt{\mathcal{J}^2 + 1}, \quad (2.3.8)$$

$$\sqrt{n^2 + (1 - \alpha)^2 \mathcal{J}^2} \pm \frac{1}{2} \sqrt{\mathcal{J}^2 + 1}. \quad (2.3.9)$$

Summary

In summary, in the flat space limit $\varepsilon = 0$, we have the following contributions from the various bosonic fields

multiplicity	field(s)	ω_n	
1	(ϕ, φ)	$\frac{\sqrt{n^2 + \mathcal{J}^2} + \sqrt{\mathcal{J}^2 + 1}}{\sqrt{n^2 + \mathcal{J}^2} - \sqrt{\mathcal{J}^2 + 1}}$	
2	U, ψ	n	(2.3.10)
2	$\beta_{\pm}, \gamma_{\pm}$	$\frac{\sqrt{n^2 + \alpha^2 \mathcal{J}^2}}{\sqrt{n^2 + (1 - \alpha)^2 \mathcal{J}^2}}$	

as well as fermionic ones

multiplicity	field(s)	ω_n	
1	Ψ	$\frac{\sqrt{n^2 + \mathcal{J}^2} + \frac{1}{2}\sqrt{\mathcal{J}^2 + 1}}{\sqrt{n^2 + \mathcal{J}^2} - \frac{1}{2}\sqrt{\mathcal{J}^2 + 1}}$ $\frac{n + \frac{1}{2}\sqrt{\mathcal{J}^2 + 1}}{n - \frac{1}{2}\sqrt{\mathcal{J}^2 + 1}}$ $\frac{\sqrt{n^2 + \alpha^2 \mathcal{J}^2} + \frac{1}{2}\sqrt{\mathcal{J}^2 + 1}}{\sqrt{n^2 + (1 - \alpha)^2 \mathcal{J}^2} - \frac{1}{2}\sqrt{\mathcal{J}^2 + 1}}$	(2.3.11)

Summing with weight $(-1)^F$ we find full cancellation of all terms. Notice that there are no ghost since we are in the static gauge.

2.4 $\mathcal{O}(\mathcal{S})$ frequencies and one-loop energy

Computing the $\mathcal{O}(\varepsilon^2 \sim \mathcal{S})$ corrections to the various frequencies we find the following results

Bosons

For the bosonic modes, we have two massless modes and 2+2 massive decoupled modes with masses $\alpha \mathcal{J}$, $(1 - \alpha) \mathcal{J}$ that do not receive corrections

$$\omega_n^{B(1,2)} = n, \quad (2.4.1)$$

$$\omega_n^{B(3,4)} = \sqrt{n^2 + \alpha^2 \mathcal{J}^2}, \quad (2.4.2)$$

$$\omega_n^{B(5,6)} = \sqrt{n^2 + (1 - \alpha)^2 \mathcal{J}^2}, \quad (2.4.3)$$

The two coupled modes give instead

$$\omega_n^{B(7,8)} = \sqrt{n^2 + \mathcal{J}^2} \pm \sqrt{\mathcal{J}^2 + 1} + \varepsilon^2 \left(\frac{1}{2\sqrt{\mathcal{J}^2 + n^2}} \pm \frac{1}{4\sqrt{\mathcal{J}^2 + 1}} \right) + \dots \quad (2.4.4)$$

Fermions

The corrections to the fermionic frequencies are

$$\omega_n^{F(1,2)} = n \pm \frac{\sqrt{\mathcal{J}^2 + 1}}{2} \pm \varepsilon^2 \frac{1}{8\sqrt{\mathcal{J}^2 + 1}} + \dots, \quad (2.4.5)$$

$$\omega_n^{F(3,4)} = \sqrt{n^2 + \mathcal{J}^2} \pm \frac{\sqrt{\mathcal{J}^2 + 1}}{2} + \varepsilon^2 \left(\frac{1}{4\sqrt{\mathcal{J}^2 + n^2}} \pm \frac{1}{8\sqrt{\mathcal{J}^2 + 1}} \right) + \dots, \quad (2.4.6)$$

$$\omega_n^{F(5,6)} = \sqrt{n^2 + \alpha^2 \mathcal{J}^2} \pm \frac{\sqrt{\mathcal{J}^2 + 1}}{2} + \varepsilon^2 \left(\frac{\alpha}{4\sqrt{\alpha^2 \mathcal{J}^2 + n^2}} \pm \frac{1}{8\sqrt{\mathcal{J}^2 + 1}} \right) + \dots, \quad (2.4.7)$$

$$\omega_n^{F(7,8)} = \sqrt{n^2 + (1-\alpha)^2 \mathcal{J}^2} \pm \frac{\sqrt{\mathcal{J}^2 + 1}}{2} + \varepsilon^2 \left(\frac{1-\alpha}{4\sqrt{(1-\alpha)^2 \mathcal{J}^2 + n^2}} \pm \frac{1}{8\sqrt{\mathcal{J}^2 + 1}} \right) + \dots \quad (2.4.8)$$

Notice that they have the general form

$$\omega_n^F = \sqrt{n^2 + \xi^2 \mathcal{J}^2} \pm \frac{\sqrt{\mathcal{J}^2 + 1}}{2} + \varepsilon^2 \left(\frac{\xi}{4\sqrt{\xi^2 \mathcal{J}^2 + n^2}} \pm \frac{1}{8\sqrt{\mathcal{J}^2 + 1}} \right) + \dots, \quad (2.4.9)$$

where $\xi = 0, 1, \alpha, 1-\alpha$. If we combine bosons and fermions for generic n we find

$$\sum_{i=1}^8 (\omega_n^{B(i)} - \omega_n^{F(i)}) = \frac{\varepsilon^2}{2} \left(\frac{1}{\sqrt{n^2 + \mathcal{J}^2}} - \frac{\alpha}{\sqrt{n^2 + \alpha^2 \mathcal{J}^2}} - \frac{1-\alpha}{\sqrt{n^2 + (1-\alpha)^2 \mathcal{J}^2}} \right) + \dots, \quad (2.4.10)$$

Low special modes

We have to take special care of low modes due to possible resonances with the σ dependent terms in the various differential operators that govern the quadratic fluctuations. This is signaled by divergent terms in the eigenfunctions or eigenvalues at special values of n . The fermionic frequencies do not have special low modes with the exception of

$$\omega_n^{F(4)} = \sqrt{n^2 + \mathcal{J}^2} - \frac{\sqrt{\mathcal{J}^2 + 1}}{2} + \varepsilon^2 \left(\frac{1}{4\sqrt{\mathcal{J}^2 + n^2}} - \frac{1}{8\sqrt{\mathcal{J}^2 + 1}} \right) + \dots, \quad (2.4.11)$$

and its analogues with α dependence. For $n = 1$ there are singularities in the wave function. Doing a more careful computation for the values $n = \pm 1$ we find that these modes mix and lead to two correction whose sum is twice the naive above value. So, in the calculation of the energy that is summed over n , we can use the above expressions. Bosons are special only in the case $\omega_{\pm 1}^{B(8)}$. These two frequencies are exactly zero at order ε^2 included.

One-loop energy at order $\mathcal{O}(\mathcal{S})$

In summary, the relevant sum of frequencies is

$$\mathfrak{S} = \varepsilon^2 \mathfrak{S}^{(2)} = \sum_{i=1}^8 (\omega_0^{B(i)} - \omega_0^{F(i)}) + 2 \left[-\omega_1^{B(8)} + \sum_{n=1}^{\infty} \sum_{i=1}^8 (\omega_n^{B(i)} - \omega_n^{F(i)}) \right], \quad (2.4.12)$$

where the generic n expressions for the ω 's have to be used and the subtraction in square brackets takes into account that the true value of $\omega_{\pm 1}^{B(8)}$ is zero, as discussed above. We can add and subtract in order to write

$$\mathfrak{S}^{(2)} = \alpha \log \alpha + (1-\alpha) \log(1-\alpha) - \frac{1}{2\sqrt{\mathcal{J}^2 + 1}} + \Delta \mathfrak{S}^{(2)}, \quad (2.4.13)$$

where

$$\Delta \mathfrak{S}^{(2)} = 2 \sum_{n=1}^{\infty} \sum_{i=1}^8 (\omega_n^{B(i)} - \omega_n^{F(i)}) - \alpha \log \alpha - (1-\alpha) \log(1-\alpha) - \frac{1}{2\mathcal{J}} = \quad (2.4.14)$$

$$\begin{aligned}
&= -\alpha \log \alpha - (1 - \alpha) \log(1 - \alpha) - \frac{1}{2\mathcal{J}} \\
&\quad + \sum_{n=1}^{\infty} \left[\frac{1}{\sqrt{n^2 + \mathcal{J}^2}} - \frac{\alpha}{\sqrt{n^2 + \alpha^2 \mathcal{J}^2}} - \frac{1 - \alpha}{\sqrt{n^2 + (1 - \alpha)^2 \mathcal{J}^2}} \right].
\end{aligned}$$

Remarkably, the quantity $\Delta\mathfrak{G}^{(2)}$ **is exponentially small** for large \mathcal{J} . The one-loop energy is obtained by dividing by 2κ . Taking into account that $\kappa = \mathcal{J} + \dots$ and that $\varepsilon^2 = \frac{2S}{\sqrt{\mathcal{J}^2 + 1}} + \dots$, we thus find

$$E_1 = \mathcal{S} \left[\frac{\alpha \log \alpha + (1 - \alpha) \log(1 - \alpha)}{\mathcal{J} \sqrt{\mathcal{J}^2 + 1}} - \frac{1}{2(\mathcal{J} + \mathcal{J}^3)} \right] + \text{exponentially suppressed at large } \mathcal{J} \quad (2.4.15)$$

The second term in square bracket is the same as in $AdS_5 \times S^5$ [33] and is the one-loop term of the exact slope function derived in [37, 38] (see also [33]). The first term can be removed by a coupling redefinition as explained in [20]. If we shift the string tension as

$$\sqrt{\lambda} \rightarrow \sqrt{\lambda} - 4\pi a, \quad (2.4.16)$$

while holding the charges S and J fixed, then the one-loop energy will get a contribution coming from the classical energy E_0 . Then choosing

$$a = \frac{\alpha \log \alpha + (1 - \alpha) \log(1 - \alpha)}{4\pi}, \quad (2.4.17)$$

we see that the α -dependent term in terms with L is removed. The shift (2.4.16) is equivalent to rewriting the string result in terms of the interpolating coupling discussed in [20] for the $SU(2)$ circular string as well as for the long folded string

$$h(\lambda) = \frac{\sqrt{\lambda}}{4\pi} + \frac{\alpha \log \alpha + (1 - \alpha) \log(1 - \alpha)}{4\pi} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \quad (2.4.18)$$

In Sec.(4), we will see that it is natural to identify $h(\lambda)$ with the Bethe Ansatz interpolating coupling.

Although we are mainly interested in string theory on $AdS_3 \times S^3 \times M^4$, we emphasize that the frequency method discussed in this section can of course be applied to the study of the short folded string in $AdS_5 \times S^5$. This application does not lead to any new results, but allows a reconciliation of the results presented in [31, 32] with the exact slope prediction [37, 38, 33]. The agreement was not possible in previous papers due to the choice $\mathcal{J} = 0$ and to the fact that the fermionic quadratic fluctuation operator has to be modified as discussed in [36, 26]. Details of the $AdS_5 \times S^5$ application are collected in App. (B).

2.5 $\mathcal{O}(S^2)$ frequencies and one-loop energy

After a long computation, we find the following non-trivial corrections

Bosons

The two coupled modes give

$$\begin{aligned}\omega_n^{B(7,8)} &= \sqrt{n^2 + \mathcal{J}^2} \pm \sqrt{\mathcal{J}^2 + 1} + \varepsilon^2 \left(\frac{1}{2\sqrt{\mathcal{J}^2 + n^2}} \pm \frac{1}{4\sqrt{\mathcal{J}^2 + 1}} \right) \\ &+ \varepsilon^4 \left(\pm \frac{-5\mathcal{J}^4 - 7\mathcal{J}^2 - (9\mathcal{J}^4 + 15\mathcal{J}^2 + 4)n^4 + 2(11\mathcal{J}^4 + 19\mathcal{J}^2 + 6)n^2}{64\mathcal{J}^2(\mathcal{J}^2 + 1)^{3/2}(n^2 - 1)^2} \right. \\ &\left. + \frac{-2n^4 + \mathcal{J}^4(-3n^4 + 5n^2 - 4) - \mathcal{J}^2(3n^6 - 3n^4 + 2n^2 + 2)}{16\mathcal{J}^2(n^2 - 1)^2(\mathcal{J}^2 + n^2)^{3/2}} \right) + \dots\end{aligned}\quad (2.5.1)$$

Fermions

The $\mathcal{O}(\varepsilon^4)$ corrected fermionic frequencies can be compactly written as

$$\omega_n^{F(1,2)} = \omega_n^F(0, \pm 1), \quad (2.5.2)$$

$$\omega_n^{F(3,4)} = \omega_n^F(1, \pm 1), \quad (2.5.3)$$

$$\omega_n^{F(5,6)} = \omega_n^F(\alpha, \pm 1), \quad (2.5.4)$$

$$\omega_n^{F(7,8)} = \omega_n^F(1 - \alpha, \pm 1), \quad (2.5.5)$$

where

$$\begin{aligned}\omega_n^F(\xi, \sigma) &= \sqrt{n^2 + \xi^2 \mathcal{J}^2} + \frac{\sigma}{2} \sqrt{\mathcal{J}^2 + 1} + \varepsilon^2 \left(\frac{\xi}{4\sqrt{\xi^2 \mathcal{J}^2 + n^2}} + \sigma \frac{1}{8\sqrt{\mathcal{J}^2 + 1}} \right) \\ &+ \varepsilon^4 \left[- \frac{\xi(2\mathcal{J}^2(2\mathcal{J}^2 + 1)\xi^6 + 3\mathcal{J}^2 n^6 + \xi n^4(3\mathcal{J}^4 \xi + \mathcal{J}^2(3 - 9\xi) - 4\xi + 3) + \xi^3 n^2(\mathcal{J}^4(2 - 9\xi) + \mathcal{J}^2 + 2\xi - 1))}{32\mathcal{J}^2(n^2 - \xi^2)^2(\mathcal{J}^2 \xi^2 + n^2)^{3/2}} \right. \\ &\left. + \sigma \frac{(-\mathcal{J}^2(5\mathcal{J}^2 + 7)\xi^4 - (9\mathcal{J}^4 + 15\mathcal{J}^2 + 4)n^4 + 2\xi n^2(\mathcal{J}^4(3\xi + 4) + \mathcal{J}^2(3\xi + 8) - 2\xi + 4))}{128\mathcal{J}^2(\mathcal{J}^2 + 1)^{3/2}(n^2 - \xi^2)^2} \right] + \dots\end{aligned}\quad (2.5.6)$$

Low special modes

The low modes can be discussed as in the previous section. In particular, we find that at this order it is still true that $\omega_{\pm 1}^{B(8)} = 0$. There is also one fermionic frequency that deserves a special analysis. It is $\omega_n^{F(4)}$ for $n = \pm 1$. The analysis of this special case shows that

$$\omega_1^{F(4)} + \omega_{-1}^{F(4)} = \sqrt{\mathcal{J}^2 + 1} + \frac{\varepsilon^2}{4\sqrt{\mathcal{J}^2 + 1}} + \frac{(-5\mathcal{J}^2 - 7)\varepsilon^4}{64(\mathcal{J}^2 + 1)^{3/2}} + \dots\quad (2.5.7)$$

All other cases are not special in any sense and the above generic- n expressions can be used.

One-loop energy at order $\mathcal{O}(\mathcal{S}^2)$

The sum over frequencies is now

$$\mathfrak{S} = \varepsilon^2 \mathfrak{S}^{(2)} + \varepsilon^4 \mathfrak{S}^{(4)} + \dots\quad (2.5.8)$$

The term $\mathfrak{S}^{(4)}$ can be further split into a part which is exponentially suppressed for large \mathcal{J} and a remainder

$$\mathfrak{S}^{(4)} = \mathfrak{S}^{(4)\text{ wrap}} + \mathfrak{S}^{(4)\text{ non-wrap}}.\quad (2.5.9)$$

The splitting is achieved by the methods illustrated in App. (C). The one-loop energy is obtained from $E_1 = \frac{\mathcal{S}}{2\kappa}$, with

$$\begin{aligned} \kappa = & \mathcal{J} + \frac{\mathcal{S}}{\mathcal{J}\sqrt{\mathcal{J}^2+1}} - \frac{(3\mathcal{J}^4+7\mathcal{J}^2+2)\mathcal{S}^2}{4(\mathcal{J}^3(\mathcal{J}^2+1)^2)} \\ & + \frac{(12\mathcal{J}^8+51\mathcal{J}^6+77\mathcal{J}^4+36\mathcal{J}^2+8)\mathcal{S}^3}{16\mathcal{J}^5(\mathcal{J}^2+1)^{7/2}} + O(\mathcal{S}^4), \end{aligned} \quad (2.5.10)$$

$$\varepsilon^2 = \frac{2\mathcal{S}}{\sqrt{\mathcal{J}^2+1}} + \frac{(\mathcal{J}^2-1)\mathcal{S}^2}{2(\mathcal{J}^2+1)^2} + \frac{(-2\mathcal{J}^4-5\mathcal{J}^2+7)\mathcal{S}^3}{8(\mathcal{J}^2+1)^{7/2}} + O(\mathcal{S}^4). \quad (2.5.11)$$

The non-wrapping part of E_1 turns out to be

$$\begin{aligned} E_1^{\text{non-wrap}} = & \left[\frac{L}{\mathcal{J}\sqrt{\mathcal{J}^2+1}} - \frac{1}{2(\mathcal{J}+\mathcal{J}^3)} \right] \mathcal{S} \\ & + \left[\left(\frac{\mathcal{J}}{2(\mathcal{J}^2+1)^2} - \frac{1}{\mathcal{J}^3} \right) L + \frac{1}{48\mathcal{J}^3(\mathcal{J}^2+1)^{5/2}} \left(6\mathcal{J}^4 + 12\pi(\mathcal{J}^2+1)^2((2\alpha-1)\cot(\pi\alpha) \right. \right. \\ & \left. \left. - \pi(\alpha-1)\alpha\csc^2(\pi\alpha)) + 45\mathcal{J}^2 - 4\pi^2(\mathcal{J}^2+1)^2 + 12(\mathcal{J}^2+1)^2\coth^{-1}(\sqrt{\mathcal{J}^2+1}) + 12 \right) \right] \mathcal{S}^2 + \dots, \end{aligned} \quad (2.5.12)$$

where $L = \alpha\log(\alpha) + (1-\alpha)\log(1-\alpha)$.

2.6 Large \mathcal{J} expansion and non-analytic contributions

Expanding at large \mathcal{J} , where $E_1 = E_1^{\text{nonwrap}}$ up to exponentially small corrections, we find

$$\begin{aligned} E_1 = & \left[\frac{L}{\mathcal{J}^2} - \frac{1}{2\mathcal{J}^3} - \frac{L}{2\mathcal{J}^4} + \frac{1}{2\mathcal{J}^5} + \frac{3L}{8\mathcal{J}^6} - \frac{1}{2\mathcal{J}^7} + \dots \right] \mathcal{S} \\ & + \left[-\frac{L}{2\mathcal{J}^3} + \frac{F(\alpha)}{\mathcal{J}^4} + \frac{\frac{1}{4}-L}{\mathcal{J}^5} + \frac{-\frac{1}{2}F(\alpha)+\frac{11}{16}}{\mathcal{J}^5} + \frac{\frac{3}{2}L-\frac{1}{6}}{\mathcal{J}^7} + \dots \right] \mathcal{S}^2 + \dots, \end{aligned} \quad (2.6.1)$$

where the function $F(\alpha)$ is

$$F(\alpha) = \frac{1}{4}\pi^2(1-\alpha)\alpha(\cot^2(\pi\alpha)+1) + \frac{1}{4}\pi(2\alpha-1)\cot(\pi\alpha) - \frac{\pi^2}{12} + \frac{1}{8}. \quad (2.6.2)$$

Remarkably, all the L -dependent terms can be removed by the previous coupling redefinition (2.4.18). In the following, we shall systematically drop them since our aim will be that of comparing the world-sheet computation with other methods based on integrability (Algebraic Curve quantization, discrete Bethe equations). In summary, the large \mathcal{J} expansion of the classical and one-loop energies can be written as

$$\begin{aligned} \mathcal{E}_0 = & \mathcal{J} + \frac{\sqrt{\mathcal{J}^2+1}}{\mathcal{J}}\mathcal{S} + \frac{(-\mathcal{J}^2-2)}{4\mathcal{J}^3(\mathcal{J}^2+1)}\mathcal{S}^2 + \dots \\ = & \mathcal{J} + \left(1 + \frac{1}{2\mathcal{J}^2} - \frac{1}{8\mathcal{J}^4} + \frac{1}{16\mathcal{J}^6} + \dots \right) \mathcal{S} + \left(-\frac{1}{4\mathcal{J}^3} - \frac{1}{4\mathcal{J}^5} + \frac{1}{4\mathcal{J}^7} + \dots \right) \mathcal{S}^2 + \dots, \end{aligned} \quad (2.6.3)$$

$$E_1 = \left(-\frac{1}{2\mathcal{J}^3} + \frac{1}{2\mathcal{J}^5} - \frac{1}{2\mathcal{J}^7} + \dots \right) \mathcal{S} + \left(\frac{F(\alpha)}{\mathcal{J}^4} + \frac{1}{4\mathcal{J}^5} + \frac{-\frac{1}{2}F(\alpha) + \frac{11}{16}}{\mathcal{J}^6} - \frac{1}{6\mathcal{J}^7} + \dots \right) \mathcal{S}^2 + \dots .$$

It is convenient to regroup the various terms appearing in the large \mathcal{J} expansion in a way that will ease the comparison with the weak-coupling expansion of the Bethe equations after re-expansion in powers of λ at fixed J as we shall discuss in in Sec. (4). Thus, we rewrite the above expansions as

$$\begin{aligned} \mathcal{E}_0 = & \left[\mathcal{J} + \left(1 + \frac{1}{2\mathcal{J}^2} \right) \mathcal{S} - \frac{1}{4\mathcal{J}^3} \mathcal{S}^2 + \dots \right]_{1L} + \left[-\frac{1}{8\mathcal{J}^4} \mathcal{S} - \frac{1}{4\mathcal{J}^5} \mathcal{S}^2 + \dots \right]_{2L} \\ & + \left[\left(\frac{1}{16\mathcal{J}^6} + \dots \right) \mathcal{S} + \left(\frac{1}{4\mathcal{J}^7} + \dots \right) \mathcal{S}^2 + \dots \right]_{HL} \end{aligned} \quad (2.6.4)$$

$$\begin{aligned} E_1 = & \left[-\frac{1}{2\mathcal{J}^3} \mathcal{S} + \frac{F(\alpha)}{\mathcal{J}^4} \mathcal{S}^2 + \dots \right]_{1L} + \left[\frac{1}{2\mathcal{J}^5} \mathcal{S} + \frac{-\frac{1}{2}F(\alpha) + \frac{11}{16}}{\mathcal{J}^6} \mathcal{S}^2 + \dots \right]_{2L} \\ & + \left[-\frac{1}{2\mathcal{J}^7} \mathcal{S} + \dots \right]_{HL} + \left[\frac{1}{4\mathcal{J}^5} \mathcal{S} - \frac{1}{6\mathcal{J}^7} \mathcal{S}^2 + \dots \right]_{non \text{ analytic}}, \end{aligned} \quad (2.6.5)$$

where

1L = one-loop in the dual CFT

2L = two-loops in the dual CFT

HL = higher-loops in the dual CFT

and, finally, the *non analytic* terms are expected to be due to the dressing phase in the ABA Bethe equations. Remarkably, these terms are not dependent on the geometrical parameter α .

For later analysis, it is convenient to write the dressing contributions together with that of the analogous terms for the folded string in $AdS_5 \times S^5$

$$\begin{aligned} E_{1, AdS_5}^{\text{dressing}} &= \left[\frac{(\mathcal{J}^2 + 2) \coth^{-1}(\sqrt{\mathcal{J}^2 + 1} + \mathcal{J})}{\mathcal{J}^3 (\mathcal{J}^2 + 1)^{3/2}} - \frac{1}{2\mathcal{J}^3 (\mathcal{J}^2 + 1)} \right] \mathcal{S}^2 + \dots \\ &= \left(\frac{0}{\mathcal{J}^5} + \frac{2}{3\mathcal{J}^7} - \frac{16}{15\mathcal{J}^9} + \dots \right) \mathcal{S}^2 + \dots, \end{aligned} \quad (2.6.6)$$

$$E_{1, AdS_3}^{\text{dressing}} = \frac{\coth^{-1}(\sqrt{\mathcal{J}^2 + 1})}{4\mathcal{J}^3 \sqrt{\mathcal{J}^2 + 1}} \mathcal{S}^2 + \dots = \left(\frac{1}{4\mathcal{J}^5} - \frac{1}{6\mathcal{J}^7} + \frac{2}{15\mathcal{J}^9} + \dots \right) \mathcal{S}^2 + \dots \quad (2.6.7)$$

According to the discussion in [35], this means that non-analytic/dressing terms start with $1/J^5 f(S/J)$ terms in AdS_5 case and $1/J^3 f(S/J)$ terms in AdS_3 case.

3 Algebraic curve quantization in $AdS_3 \times S^3 \times T^4$

In this section we focus on the special limit case of string propagation on $AdS_3 \times S^3 \times T^4$. This is formally obtained from the general case by taking the singular limit $\alpha \rightarrow 1$. The motivation behind the study of this special case is that the (most interesting) dressing contributions are apparently independent on α (see (2.6.5)) and we expect to clarify their origin in the simpler

T^4 setup. In particular, we describe in this section the quantization of the classical algebraic curve relevant to $AdS_3 \times S^3 \times T^4$ by the methods of [39]. We shall show that the frequencies obtained in the previous section are precisely recovered by this method and we shall discuss the origin of dressing from the usual unit circle contribution that gives the familiar Hernandez-Lopez phase in $AdS_5 \times S^5$.

3.1 Classical data

The classical data is in terms of a two cut curve parametrized by the cut endpoints a, b related to the classical charges and energy by the same equations as in $AdS_5 \times S^5$

$$\mathcal{S} = \frac{ab+1}{2\pi ab} \left[b \mathbb{E} \left(1 - \frac{a^2}{b^2} \right) - a \mathbb{K} \left(1 - \frac{a^2}{b^2} \right) \right], \quad (3.1.1)$$

$$\mathcal{J} = \frac{1}{\pi b} \sqrt{(a^2-1)(b^2-1)} \mathbb{K} \left(1 - \frac{a^2}{b^2} \right), \quad (3.1.2)$$

$$\mathcal{E} = \frac{ab-1}{2\pi ab} \left[b \mathbb{E} \left(1 - \frac{a^2}{b^2} \right) + a \mathbb{K} \left(1 - \frac{a^2}{b^2} \right) \right]. \quad (3.1.3)$$

The relevant quasi-momenta $\tilde{p}_{1,2,3,4}$ and $\hat{p}_{1,2,3,4}$ are the same as in $AdS_5 \times S^5$ and can be written in terms of

$$\tilde{p}_2 = \frac{2\pi \mathcal{J} x}{x^2 - 1}, \quad (3.1.4)$$

$$\begin{aligned} \hat{p}_2 = & \pi - 2\pi \mathcal{J} \left(\frac{a}{a^2-1} - \frac{x}{x^2-1} \right) \sqrt{\frac{(a^2-1)(b^2-x^2)}{(b^2-1)(a^2-x^2)}} \\ & + \frac{8i\pi ab \mathcal{S}}{(b-a)(ab+1)} \mathbb{F} \left(i \operatorname{arcsinh} \left(\sqrt{-\frac{(a-b)(a-x)}{(a+b)(a+x)}} \right), \frac{(a+b)^2}{(a-b)^2} \right) \\ & + \frac{2i\pi(a-b)\mathcal{J}}{\sqrt{(a^2-1)(b^2-1)}} \mathbb{E} \left(i \operatorname{arcsinh} \left(\sqrt{-\frac{(a-b)(a-x)}{(a+b)(a+x)}} \right), \frac{(a+b)^2}{(a-b)^2} \right), \end{aligned} \quad (3.1.5)$$

as

$$\tilde{p}_1 = -\tilde{p}_3 = -\tilde{p}_4 = \tilde{p}_2, \quad -\hat{p}_1(1/x) = -\hat{p}_3(x) = \hat{p}_4(1/x) = \hat{p}_2(x). \quad (3.1.6)$$

3.2 Algebraic curve one-loop quantization and comparison with world-sheet

The relevant string polarizations can be assigned at the two sheets A, B of the disconnected AdS_3 algebraic curve (see for instance [20]) according to the following table where we adopt

the $AdS_5 \times S^5$ labeling of off-shell frequencies of [40]

polarization	sheet(s)	off-shell frequency
$(\tilde{2}, \tilde{3})$	$A + B$	$(1 + 1) \times \Omega_S$
$(\hat{1}, \hat{4})$	B	Ω_1
$(\hat{2}, \hat{3})$	A	Ω_A
$(\hat{2}, \tilde{4})$	A	$-2 \times \Omega_3,$
$(\tilde{2}, \hat{4})$	B	$-2 \times \Omega_4$

(3.2.1)

where the off-shell frequencies are

$$\Omega_S(x) = \frac{2}{ab-1} \frac{\sqrt{a^2-1}\sqrt{b^2-1}}{x^2-1}, \quad (3.2.2)$$

$$\Omega_A(x) = \frac{2}{ab-1} \left(1 - \frac{\sqrt{x-a}\sqrt{x+a}\sqrt{x-b}\sqrt{x+b}}{x^2-1} \right), \quad (3.2.3)$$

$$\Omega_1(x) = -2 - \Omega_A(1/x), \quad (3.2.4)$$

$$\Omega_3(x) = \frac{1}{2}(\Omega_A(x) + \Omega_S(x)), \quad (3.2.5)$$

$$\Omega_4(x) = \frac{1}{2}(\Omega_S(x) - \Omega_A(1/x)) - 1. \quad (3.2.6)$$

For each polarization (I, J) , we compute $x_n^{(I,J)}$ from

$$p^I(x_n^{(I,J)}) - p^J(x_n^{(I,J)}) = 2\pi n, \quad (3.2.7)$$

and plug it into the relevant off-shell frequency. Once this is done, we find the following relations with the world-sheet frequencies

$$\kappa \Omega_S(x_n^{(\tilde{2}, \tilde{3})}) = \sqrt{n^2 + \mathcal{J}^2} + \Delta\Omega_S, \quad (3.2.8)$$

$$\kappa \Omega_1(x_n^{(\hat{1}, \hat{4})}) = \omega_n^{B(7)} + \Delta\Omega_1, \quad (3.2.9)$$

$$\kappa \Omega_A(x_n^{(\tilde{2}, \tilde{3})}) = \omega_n^{B(8)} + \Delta\Omega_A, \quad (3.2.10)$$

$$\kappa \Omega_3(x_n^{(\hat{2}, \hat{4})}) = \omega_n^F(1, -1) + \Delta\Omega_3, \quad (3.2.11)$$

$$\kappa \Omega_4(x_n^{(\tilde{2}, \tilde{4})}) = \omega_n^F(1, 1) + \Delta\Omega_4, \quad (3.2.12)$$

where the shifts $\Delta\Omega$ are collected in App. (D). These shifts are independent on n and separately cancel on each sheet:

$$\Delta\Omega_S + \Delta\Omega_1 - 2\Delta\Omega_4 = \mathcal{O}(\mathcal{S}^3), \quad (3.2.13)$$

$$\Delta\Omega_S + \Delta\Omega_A - 2\Delta\Omega_3 = \mathcal{O}(\mathcal{S}^3), \quad (3.2.14)$$

thus proving the complete equivalence between the world-sheet and quantized algebraic curve computation of the frequencies.

3.3 Dressing from the algebraic curve

The one-loop energy can be computed in the so-called algebraic curve regularization by transforming the sum over frequencies into a contour integral and deforming the contour along the unit circumference plus additional cut contributions (see for instance the detailed discussion in [24]). The unit circumference contour is expected to give the dressing contribution. Let us denote by A the various string polarizations, label (A_1, A_2) the associated pair of quasimomenta and define $(-1)^{F_A}$ to be the Bose-Fermi sign taking into account statistics. The relevant formula reads

$$E_1^{\text{dressing}} = \frac{1}{4\pi} \int_{-1}^1 dx \sum_A (-1)^{F_A} \Omega^{(A)}(x) (p_{A_1}(x) - p_{A_2}(x))', \quad (3.3.1)$$

where the integral is computed along the upper unit half-circumference. Expanding in powers of \mathcal{S} , we find

$$\begin{aligned} E_1^{\text{dressing}} &= \mathcal{S}^2 \int_{-1}^1 dx \frac{8 \left(4\mathcal{J}^4 x^6 + \frac{1}{4} (x^2 - 1)^3 (x^2 + 1) + \mathcal{J}^2 (x^6 + 3x^4 - 3x^2 - 1) x^2 \right)}{(4\mathcal{J}^3 x^2 - \mathcal{J} (x^2 - 1)^2)^3} + \mathcal{O}(\mathcal{S}^3) \\ &= \mathcal{S}^2 \left[\frac{\coth^{-1}(\sqrt{\mathcal{J}^2 + 1})}{4\mathcal{J}^3 \sqrt{\mathcal{J}^2 + 1}} + \frac{1}{2\mathcal{J}^5} \right] + \mathcal{O}(\mathcal{S}^3). \end{aligned} \quad (3.3.2)$$

Comparing with (2.6.7), we see that there is an extra piece $\frac{1}{2\mathcal{J}^5}$. This is due to a regularization issue that we now explain.

3.4 Matching of AC and WS regularizations

It is known that there can be a regularization mismatch between the standard world-sheet computation and the algebraic curve one. The reason is simple. The one-loop energy is evaluated as a sum over frequencies ω_n^A where A labels the various modes. The world-sheet regularization amounts to summing over n with a cut-off $|n| \leq N$ independent on A and taking the finite limit $N \rightarrow \infty$. Instead, in the algebraic curve regularization, one takes a fixed spectral radius cut-off $|x - 1| \geq 1 + \varepsilon$ and sends $\varepsilon \rightarrow 0$. This cut-off is effectively equivalent to different mode number cut-offs N_A and there is a mismatch between the two procedures. The mismatch can be computed as follows (see [20] for the example of a circular string classical solution).

Let us have in mind the example of the short folded string, but try to be as general as possible. Setting

$$x = 1 + \varepsilon \mathcal{J} + \frac{\varepsilon}{2} \mathcal{J}^2, \quad (3.4.1)$$

we have

$$N_A(x) = \frac{p_{A_1}(x) - p_{A_2}(x)}{2\pi} = \frac{1}{\varepsilon} + \Delta_A^{(0)} + \varepsilon \Delta_A^{(1)} + \dots, \quad (3.4.2)$$

$$\kappa \omega_A(x) = \frac{1}{\varepsilon} + \Omega_A^{(0)} + \varepsilon \Omega_A^{(1)} + \dots, \quad (3.4.3)$$

where $A = (A_1, A_2)$ are the string polarizations. Inverting the relation between ε and N_A :

$$\varepsilon = \frac{1}{N_A} + \frac{\Delta_A^{(0)}}{N_A^2} + \frac{\Delta_A^{(0)2} + \Delta_A^{(1)}}{N_A^3} + \dots, \quad (3.4.4)$$

we can write the large mode number expansion of each world-sheet frequency

$$\kappa \Omega_A = N_A + (\Omega_A^{(0)} - \Delta_A^{(0)}) + \frac{\Omega_A^{(1)} - \Delta_A^{(1)}}{N_A} + \dots \quad (3.4.5)$$

Let C_A be the numerical integers that are used to combine the various modes. We know that $\sum_A C_A N_A = 0$, so

$$\sum_A C_A = \sum_A C_A \Delta_A^{(0)} = \sum_A C_A \Delta_A^{(1)} = 0. \quad (3.4.6)$$

Also, UV convergence requires

$$\sum_A C_A (\Omega_A^{(0)} - \Delta_A^{(0)}) = \sum_A C_A (\Omega_A^{(1)} - \Delta_A^{(1)}) = 0, \quad (3.4.7)$$

or, due to the previous relation

$$\sum_A C_A \Omega_A^{(0)} = \sum_A C_A \Omega_A^{(1)} = 0. \quad (3.4.8)$$

These relations actually hold for each of the two sheets separately. The mismatch due to regularization is [20]

$$\begin{aligned} \kappa \delta^{\text{AC-WS}} &= \lim_{\varepsilon \rightarrow 0} \sum_A C_A \left[\frac{1}{2} N^2 + (\Omega_A^{(0)} - \Delta_A^{(0)}) N \right]_{\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon} + \Delta_A^{(0)} + \varepsilon \Delta_A^{(1)} + \dots} = \\ &= \sum_A C_A \left[\Delta_A^{(1)} - \frac{1}{2} \Delta_A^{(0)2} + \Omega_A^{(0)} \Delta_A^{(0)} \right], \end{aligned} \quad (3.4.9)$$

and, using again the sum rules

$$\delta^{\text{AC-WS}} = \frac{1}{\kappa} \sum_A C_A \left[-\frac{1}{2} \Delta_A^{(0)2} + \Omega_A^{(0)} \Delta_A^{(0)} \right]. \quad (3.4.10)$$

For the folded string in the short limit we report the values of $\Delta^{(0)}$ and $\Omega^{(0)}$ in App. (E)). We find

$$\delta^{\text{AC-WS}} = \frac{\mathcal{S}^2}{2 \mathcal{J}^5} + \mathcal{O}(\mathcal{S}^3), \quad (3.4.11)$$

explaining the extra term in (3.3.2).

4 Large J limit of the ABA Bethe equations

In [13], a set of all-loop asymptotic Bethe equations (ABA) has been proposed to describe strings on $AdS_3 \times S^3 \times S^1$, with symmetry $d(2, 1; \alpha)^2$, valid for all values of α . In the spirit of [35], it is interesting to compare the large J and fixed S expansion of the Bethe Ansatz energy at weak coupling one and two-loops level with the large \mathcal{J} expansion of the classical and one-loop string energy. The aim of the comparison is that of checking whether weak-coupling contributions of the form $\frac{S^p}{J^q} \lambda^n$ match the analogous terms $\frac{S^p}{J^q} \lambda^{n+\frac{p-q}{2}}$ in the string theory. Such a matching would show that these terms are protected from possible non-trivial effects that could appear in the extrapolation between weak and strong coupling.

In particular, one can expect that these non-trivial effects are related to the dressing phase(s) in the Bethe equations and do not modify the above contributions at least for low integer values of n , *i.e.* one and two loops contributions have a chance of being equal to the corresponding terms in the string theory. For these reasons, we shall first analyze the ABA equations without introducing any dressing phase(s) and in a second step shall discuss the role of dressing.

4.1 ABA equations without dressing

The relevant subset of Bethe equations involves roots at two coupled nodes. In the notation of [13] the logarithmic form of the equations suitable for the analysis of the folded string solution can be written

$$J \log \left(\frac{x_{1,i}^+}{x_{1,i}^-} \right) = \sum_{k \neq i}^S \log \left(\frac{1 - \frac{h}{x_{1,i}^+ x_{1,k}^-}}{1 - \frac{h}{x_{1,i}^- x_{1,k}^+}} \sigma_1^2(x_{1,i}, x_{1,k}) \right) + \sum_{k=1}^S \log \left(\frac{x_{1,i}^- - x_{3,k}^+}{x_{1,i}^+ - x_{3,k}^-} \right) + 2\pi i n_i, \quad (4.1.1)$$

$$J \log \left(\frac{x_{3,i}^+}{x_{3,i}^-} \right) = \sum_{k \neq i}^S \log \left(\frac{1 - \frac{h}{x_{3,i}^+ x_{3,k}^-}}{1 - \frac{h}{x_{3,i}^- x_{3,k}^+}} \sigma_3^2(x_{3,i}, x_{3,k}) \right) + \sum_{k=1}^S \log \left(\frac{x_{3,i}^- - x_{1,k}^+}{x_{3,i}^+ - x_{1,k}^-} \right). \quad (4.1.2)$$

Here, the quantities $x_{\ell,i}^\pm$, $\ell = 1, 3$, $i = 1, \dots, S$, are defined on the nodes 1, 3 as

$$x_{1,i}^\pm = x \left(u_{1,i} \pm i \alpha \right), \quad x_{3,i}^\pm = x \left(u_{3,i} \pm i (1 - \alpha) \right), \quad (4.1.3)$$

where

$$x(u) = \frac{u}{2} \left(1 + \sqrt{1 - \frac{4h}{u^2}} \right). \quad (4.1.4)$$

The coupling $h(\lambda)$ is an interpolating coupling with the strong coupling expansion at $\lambda \gg 1$ $h(\lambda) = \frac{\sqrt{\lambda}}{2\pi} + \mathcal{O}(1)$. The energy and momentum are computed from

$$E(S, J) = i h \sum_{\ell=1,3} \sum_{k=1}^S \left(\frac{1}{x_{\ell,k}^+} - \frac{1}{x_{\ell,k}^-} \right), \quad (4.1.5)$$

$$e^{iP} = \prod_{\ell=1,3} \prod_k \frac{x_{\ell,k}^+}{x_{\ell,k}^-} = 1. \quad (4.1.6)$$

We shall write the weak-coupling expansion of the energy as

$$E(S, J) = 4\pi^2 h E^{(1)}(S, J) + (4\pi^2 h)^2 E^{(2)}(S, J) + \dots, \quad (4.1.7)$$

where the choice of normalization of $E^{(n)}$ will be explained later.

4.1.1 Large J expansion of the one-loop energy

The one-loop Bethe roots $u_{\ell,i}$ of the folded string are symmetric under $u \rightarrow -u$

$$u_{1,i} = (U_1, U_2, \dots, U_{\frac{S}{2}}, -U_1, -U_2, \dots, -U_{\frac{S}{2}}), \quad (4.1.8)$$

$$u_{3,i} = (U_{\frac{S}{2}+1}, U_{\frac{S}{2}+2}, \dots, U_S, -U_{\frac{S}{2}+1}, -U_{\frac{S}{2}+2}, \dots, -U_S), \quad (4.1.9)$$

and the mode numbers are ⁶

$$n_i = (-1, \dots, -1, 1, \dots, 1). \quad (4.1.10)$$

As we said, we first consider the ABA without dressing and put $\sigma_{1,3} \rightarrow 1$.

The numerical analysis of the large J Bethe roots suggests the following Ansatz⁷

$$u_i = -\frac{J}{\pi} + c_i^{(1)} \sqrt{J} + c_i^{(2)} + c_i^{(3)} \frac{1}{\sqrt{J}} + c_i^{(4)} \frac{1}{J} + \dots, \quad i = 1, \dots, \frac{S}{2}, \quad (4.1.11)$$

$$u_i = -\frac{J}{\pi} + c_{i-\frac{S}{2}}^{(1)} \sqrt{J} + c_i^{(2)} + c_i^{(3)} \frac{1}{\sqrt{J}} + c_i^{(4)} \frac{1}{J} + \dots, \quad i = \frac{S}{2} + 1, \dots, S. \quad (4.1.12)$$

The correlation between the coefficients $c_i^{(1)}$ is a non trivial remark. Expanding the Bethe equations we find the following remarkable relation

$$\pi^2 c_i^{(1)} = 2 \sum_{j \neq i} \frac{1}{c_i^{(1)} - c_j^{(1)}}, \quad i = 1, \dots, \frac{S}{2}. \quad (4.1.13)$$

It implies that the coefficients $c_i^{(1)}$ for $i = 1, \dots, \frac{S}{2}$ are the roots of

$$H_{\frac{S}{2}} \left(\frac{\pi}{\sqrt{2}} c^{(1)} \right) = 0, \quad (4.1.14)$$

where H_n is the n -th Hermite polynomial. The other coefficients obey some analytical relations like (here again $i = 1, \dots, \frac{S}{2}$)

$$c_{i+\frac{S}{2}}^{(2)} = c_i^{(2)} - \cot(\pi\alpha), \quad (4.1.15)$$

⁶Notice that the mode numbers of the 3-roots are trivial with the standard branch of the logarithm.

⁷A posteriori, the expansion based on this Ansatz will match perfectly a numerical fit to the various $1/J$ coefficients of the numerical Bethe roots.

$$c_{i+\frac{S}{2}}^{(3)} = c_i^{(3)} + \frac{\pi^2(2\alpha-1)}{2\sin^2(\pi\alpha)} c_i^{(1)}, \quad (4.1.16)$$

but basically must be determined numerically keeping α generic. Plugging the resulting expansion of the Bethe roots in the expression for the energy (4.1.5) we obtain the following expression for the first three orders of the large J expansion of the one-loop energy $E^{(1)}$

$$E^{(1)}(S, J) = \frac{S}{2J^2} - \left(\frac{S^2}{4} + \frac{S}{2} \right) \frac{1}{J^3} + \left[\frac{3}{16} S^3 + F(\alpha) S^2 + \frac{S}{2} \right] \frac{1}{J^4} + \dots, \quad (4.1.17)$$

where $F(\alpha)$ has been defined in (2.6.2).

4.2 Large J expansion at two-loops

Expanding at order $\mathcal{O}(h^2)$ the Bethe roots are written as

$$u_i = u_i^{(0)} + h u_i^{(1)} + \dots, \quad (4.2.1)$$

where $u_i^{(0)}$ are given in (4.1.11) and (4.1.12) and

$$u_i^{(1)} = -\frac{2\pi}{J} + \frac{d_i^{(1)}}{J^{3/2}} + \frac{d_i^{(2)}}{J^2} + \frac{d_i^{(3)}}{J^{5/2}} + \frac{d_i^{(4)}}{J^3} + \dots, \quad i = 1, \dots, \frac{S}{2}, \quad (4.2.2)$$

$$u_i^{(1)} = -\frac{2\pi}{J} + \frac{d_{i-\frac{S}{2}}^{(1)}}{J^{3/2}} + \frac{d_{i-\frac{S}{2}}^{(2)}}{J^2} + \frac{d_{i-\frac{S}{2}}^{(3)}}{J^{5/2}} + \frac{d_{i-\frac{S}{2}}^{(4)}}{J^3} + \dots, \quad i = \frac{S}{2} + 1, \dots, S. \quad (4.2.3)$$

Again, the correlation between the coefficients $d_i^{(1)}$ and $d_i^{(2)}$ is a non trivial remark. Evaluating the two-loop energy we find

$$E^{(2)} = -\frac{S}{8J^4} + \left(-\frac{S^2}{4} + \frac{S}{2} \right) \frac{1}{J^5} + \left[\frac{11}{32} S^3 + \left(-\frac{1}{2} F(\alpha) + \frac{11}{16} \right) S^2 - \frac{11}{8} S \right] \frac{1}{J^6} + \dots \quad (4.2.4)$$

4.2.1 Comparison with string theory

If we assume the following weak-coupling expansion of the interpolating coupling $h(\lambda)$

$$h(\lambda) = \frac{\lambda}{4\pi^2} + \mathcal{O}(\lambda^2), \quad (4.2.5)$$

then, our one and two-loop results (4.1.17, 4.2.4) nicely reproduce the $1L$ and $2L$ terms in (2.6.4, 2.6.5). The HL terms would require a three or higher-loop computation at weak-coupling but are expected to violate matching being unprotected. The most interesting point of the comparison is the fact that the *non analytic* terms in (2.6.5) are clearly absent. They require the introduction of dressing phases to which we devote the next section.

5 ABA with dressing at $\alpha = 1$

We discuss dressing in the ABA for the $\alpha = 1$ case of string on $AdS_3 \times S^3 \times T^4$ due to several reasons

- a) We already observed that dressing effects computed by world-sheet methods are independent on α , so this should be a simplification that does not spoil any essential feature.
- b) We want to build on the algebraic curve detailed description that we have presented in this case.
- c) We want to make contact with the discussion of semiclassical dressing discussed in [20].

5.1 Prediction according to the BLMT proposal

The relevant ABA has been proposed in [4] and its reduction to the $\mathfrak{sl}(2)$ sector gives the same Bethe equations as in $AdS_5 \times S^5$ although with a possibly different phase. They read

$$\left(\frac{x_i^+}{x_i^-}\right)^J = \prod_{j \neq i}^S \frac{x_i^- - x_j^+}{x_i^+ - x_j^-} \frac{1 - \frac{1}{x_i^+ x_j^-}}{1 - \frac{1}{x_i^- x_j^+}} e^{2i\theta_{ij}}, \quad (5.1.1)$$

where the phase is

$$\theta_{ij} = \sum_{r,s \geq 1} \left[h c_{r,s}^{(0)} + c_{r,s}^{(1)} + \dots \right] q_s(u_i) q_r(u_j). \quad (5.1.2)$$

in terms of the higher charges

$$q_r = \frac{i}{r-1} \left(\frac{1}{(x^+)^{r-1}} - \frac{1}{(x^-)^{r-1}} \right). \quad (5.1.3)$$

Notice that for these $\alpha = 1$ Bethe equations the map $x(u)$ is basically the same as in $AdS_5 \times S^5$

$$x + \frac{1}{x} = \frac{u}{h}, \quad x^\pm + \frac{1}{x^\pm} = \frac{u \pm \frac{i}{2}}{h}. \quad (5.1.4)$$

and the energy has a factor 2 fixed by the $\alpha = 1$ dispersion relations

$$E_1 = 2i h \sum_{i=1}^S \left(\frac{1}{x_i^+} - \frac{1}{x_i^-} \right). \quad (5.1.5)$$

We now expand E_1 at large J and extract the dressing contribution which is \mathcal{S}^2 times a series with odd powers of $1/\mathcal{J}$. To do so, we have to fix the relation between $\sqrt{\lambda}$ and h at strong coupling that we take equal to the $AdS_5 \times S^5$ one, i.e. $h = \frac{\sqrt{\lambda}}{4\pi} + \dots$. Then, we find

$$E_1^{\text{dressing}} = \left(\frac{a_{12}}{4\mathcal{J}^5} + \frac{-3a_{12} + a_{14} - a_{23}}{16\mathcal{J}^7} \right)$$

$$+\frac{-10 a_{12} + 5 a_{14} - a_{16} + 5 a_{23} - a_{25} + a_{34}}{64 \mathcal{J}^9} + \dots \Big) \mathcal{S}^2 + \dots, \quad (5.1.6)$$

where

$$a_{r,s} = c_{r,s}^{(1)} - c_{s,r}^{(1)}. \quad (5.1.7)$$

If we plug in this expression the Hernandez-Lopez values [22] valid for the $AdS_5 \times S^5$ case

$$c_{r,s}^{(1)} = -8 \frac{1 - (-1)^{r+s}}{2} \frac{(r-1)(s-1)}{(r+s-2)(s-r)}, \quad (5.1.8)$$

we recover the expansion in (2.6.6). If instead, we use the BLMT coefficients that have been proposed in [20], i.e.

$$c_{r,s}^{(1)} = 2 \frac{1 - (-1)^{r+s}}{2} \frac{s-r}{r+s-2}, \quad (5.1.9)$$

we find

$$\begin{aligned} E_1^{\text{dressing}} &= \left(\frac{1}{\mathcal{J}^5} - \frac{7}{12 \mathcal{J}^7} + \frac{109}{240 \mathcal{J}^9} + \dots \right) \mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3) \\ &= \left[\frac{\coth^{-1}(\sqrt{\mathcal{J}^2 + 1})}{2 \mathcal{J}^3 \sqrt{\mathcal{J}^2 + 1}} + \frac{1}{2 \mathcal{J}^4 \sqrt{\mathcal{J}^2 + 1}} \right] \mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3). \end{aligned} \quad (5.1.10)$$

The analytic expression resumming the three terms of the large \mathcal{J} series is not a mere conjecture. Indeed, it will be strongly motivated and explained in the next section devoted to a discussion.

5.2 Discussion

The classical scaling limit of (5.1.1) can be written as

$$2\pi n + \frac{4\pi \mathcal{J} x}{x^2 - 1} = 2H(x) - 2\frac{G(0)}{x^2 - 1}, \quad (5.2.1)$$

where the functions H and G are defined as

$$G(x) = \sum_{k=1}^S \frac{\hat{\alpha}(x_k)}{x - x_k}, \quad H(x) = \sum_{k=1}^S \frac{\hat{\alpha}(x)}{x - x_k}, \quad \hat{\alpha}(x) = \frac{1}{h} \frac{x^2}{x^2 - 1}. \quad (5.2.2)$$

The analysis of [20] shows that one-loop semiclassical effects associated with the unit circumference in the spectral plane are incorporated by adding to the r.h.s. of this equation the potential term

$$\mathcal{V}(x) = \int_{-1}^1 \frac{dy}{2\pi} \left[\partial_y G(y) \frac{\hat{\alpha}(x)}{x - y} + \partial_y G(1/y) \frac{\hat{\alpha}(1/x)}{1/x - y} \right], \quad (5.2.3)$$

where the notation is $\int_{-1}^1 = \frac{1}{2} \int_{C^+} + \frac{1}{2} \int_{C^-}$ and the half circumferences C^\pm (and their orientation) are defined in the caption of figure 4 of [24]. This term cannot be interpreted as a phase. This is possible up to a remainder if we integrate by parts the second term in the integral

$$\mathcal{V}(x) = \mathcal{V}_{\text{phase}}(x) + \Delta \mathcal{V}(x), \quad (5.2.4)$$

where

$$\mathcal{V}_{\text{phase}}(x) = \int_{-1}^1 \frac{dy}{2\pi} \left[G'(y) \frac{\hat{\alpha}(x)}{x-y} - G(1/y) \left(\frac{\hat{\alpha}(1/x)}{1/x-y} \right)' \right], \quad (5.2.5)$$

$$\Delta\mathcal{V}(x) = \frac{\hat{\alpha}(1/x)}{2\pi} \left[\frac{G(1)}{1/x-1} - \frac{G(-1)}{1/x+1} \right]. \quad (5.2.6)$$

The role of $\Delta\mathcal{V}$ is unclear at the moment. A possible interpretation of this term is that of a modification that it is necessary to introduce at the level of the discrete Bethe equations (5.1.1) going beyond the recipes described in [6] and necessary in order to match semiclassical string theory at one-loop. Hints at such non-trivial modifications are also present in the recent analyses in [27] and [18].

What can be remarked here, is that the phase term $\mathcal{V}_{\text{phase}}(x)$ is fully consistent with the result (5.1.10). Indeed, the integration by parts (and neglecting the controversial term $\Delta\mathcal{V}$) amounts to make the same transformation in (3.3.1). In the notation of Sec. (3.4), integration by parts on a subset \mathcal{A} of polarizations gives the contribution

$$\text{IBP}_{\mathcal{A}} = \lim_{x \rightarrow 1} \sum_{A \in \mathcal{A}} C_A \omega_A(x) N_A(x) = \frac{1}{\kappa} \sum_{A \in \mathcal{A}} C_A \left[\Delta_A^{(1)} + \Omega_A^{(1)} + \Omega_A^{(0)} \Delta_A^{(0)} \right]. \quad (5.2.7)$$

If \mathcal{A} is the second sheet, then we simply have

$$\text{IBP}_{\text{second sheet}} = \frac{1}{\kappa} \sum_{A \in \text{second sheet}} C_A \Omega_A^{(0)} \Delta_A^{(0)}. \quad (5.2.8)$$

Evaluating this quantity according to the results in App. (E), we find

$$\text{IBP}_{\text{second sheet}} = \left(\frac{1}{2\mathcal{J}^5} - \frac{1}{2\mathcal{J}^4 \sqrt{1+\mathcal{J}^2}} \right) \mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3). \quad (5.2.9)$$

This is precisely the piece that we have to add to (5.1.10) in order to recover the old AC regularized result equivalent, up to regularization correction, to the world-sheet computation⁸.

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A Simplification of the quadratic fermionic fluctuation operator

In this appendix, we explain how to deal with the fermionic operator (2.2.11). It is convenient to set

$$f_1(\sigma) = \frac{\kappa w \mathcal{J}}{2(\rho'^2 + \mathcal{J}^2)}, \quad f_2(\sigma) = \frac{\rho'}{2}, \quad f_3(\sigma) = \frac{\sqrt{\rho'^2 + \mathcal{J}^2} - \rho'}{2}. \quad (\text{A.1})$$

⁸ To be more precise, that there is also a global factor of 2 compared to the $0 < \alpha < 1$. This is known to be related to the extra massless modes that are present at $\alpha = 1$ as discussed in [20] for a circular string solution as well as for the long folded string.

Squaring \mathcal{D}_F we find

$$\begin{aligned}
-\mathcal{D}_F^2 = & -\partial^2 + f_1^2 + 2\alpha(\alpha-1)f_3(2f_2+f_3) + \Gamma_{02} \left[f_2' + 2f_2\partial_\sigma + f_3' + 2f_3\partial_\sigma \right] \\
& + \Gamma_{12} \left[-2(f_2+f_3)\partial_\tau \right] + \Gamma_{25} \left[\sqrt{\alpha}(f_1' + 2f_1\partial_\sigma) \right] + \Gamma_{28} \left[\sqrt{1-\alpha}(f_1' + 2f_1\partial_\sigma) \right] \\
& + \Gamma_{1234} \left[-2\alpha f_1(f_2+f_3) \right] + \Gamma_{1267} \left[-2(1-\alpha)f_1(f_2+f_3) \right] \\
& + \Gamma_{1345} \left[\sqrt{\alpha}(f_2' + \alpha f_3') \right] + \Gamma_{1348} \left[\alpha\sqrt{1-\alpha}f_3' \right] + \Gamma_{1567} \left[(1-\alpha)\sqrt{\alpha}f_3' \right] \\
& + \Gamma_{1678} \left[\sqrt{1-\alpha}(f_2' + (1-\alpha)f_3') \right] + \Gamma_{3467} \left[-2\alpha(1-\alpha)f_3(2f_2+f_3) \right].
\end{aligned}$$

At this point it is convenient to use an explicit representation of the 10d Γ matrices. We start from the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \epsilon = i\sigma_2. \quad (\text{A.2})$$

and build the Spin(8) Clifford algebra as

$$\begin{aligned}
\gamma^1 &= \epsilon \otimes \epsilon \otimes \epsilon & \gamma^2 &= 1 \otimes \sigma_1 \otimes \epsilon \\
\gamma^3 &= 1 \otimes \sigma_3 \otimes \epsilon & \gamma^4 &= \sigma_1 \otimes \epsilon \otimes 1 \\
\gamma^5 &= \sigma_3 \otimes \epsilon \otimes 1 & \gamma^6 &= \epsilon \otimes 1 \otimes \sigma_1 \\
\gamma^7 &= \epsilon \otimes 1 \otimes \sigma_3 & \gamma^8 &= 1 \otimes 1 \otimes 1
\end{aligned} \quad (\text{A.3})$$

Then, we define the 16×16 matrices

$$\begin{aligned}
\gamma_{16}^A &= \begin{pmatrix} 0 & \gamma^A \\ (\gamma^A)^T & 0 \end{pmatrix} & \gamma_{16}^0 &= 1_{16} & \gamma_{16}^9 &= \gamma_{16}^{12345678} \\
\gamma^\mu &= \{1, \gamma^A, \gamma^9\}_{16} & \bar{\gamma}^\mu &= \{-1, \gamma^A, \gamma^9\}_{16}
\end{aligned} \quad (\text{A.4})$$

and finally the 32×32 10d Dirac matrices according to

$$\Gamma_{\{\mu\}} = \begin{pmatrix} 0 & \gamma^\mu \\ \bar{\gamma}^\mu & 0 \end{pmatrix}. \quad (\text{A.5})$$

Our main observation is that we can find a relatively simple invertible matrix U such that

$$U \Gamma_{ABC\dots} U^{-1}, \quad (\text{A.6})$$

is composed of eight 4×4 blocks on the diagonal for all 11 Gamma matrix structures appearing in \mathcal{D}_F^2 . This matrix is obtained from the eigenvectors of the commuting subset of the 11 structures and reads

$$U = \begin{pmatrix} \mathcal{U}_{1,1} & \mathcal{U}_{1,2} \\ \mathcal{U}_{2,1} & \mathcal{U}_{2,2} \end{pmatrix} \quad (\text{A.7})$$

where

[illegible]

$$\mathcal{U}_{1,2} = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -i & i & -1 & -i & 1 & 1 & i \end{vmatrix} \quad (\text{A.9})$$

[illegible]

$$\mathcal{U}_{2,2} = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i & i & 1 & i & 1 & 1 & -i \end{vmatrix} \quad (\text{A.11})$$

The 8 explicit 4×4 matrices are definitely tractable although they have a rather complicated form that we do not write in explicit form.

B Short string limit for the folded string in $AdS_5 \times S^5$

This appendix is devoted to the calculation of $\mathcal{O}(\mathcal{S})$ frequencies for the folded string in AdS_5 . As we mentioned in the main text, the aim of this application is to show that working at finite $\mathcal{J} > 0$ with the correct fermionic fluctuation operator [36, 26] gives full agreement with the exact slope result derived by integrability methods [37, 38, 33]. The classical solution is derived in [30] and has the same form as in AdS_3 .

B.1 One-loop (quadratic) fluctuations

Expanding the $AdS_5 \times S^5$ superstring action in conformal gauge to quadratic order in the fluctuations near the folded spinning string solution one finds

$$\widetilde{\mathcal{S}} = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau \int_0^{2\pi} d\sigma (\widetilde{\mathcal{L}}_B + \widetilde{\mathcal{L}}_F), \quad (\text{B.1.1})$$

where the fluctuation lagrangians are separately discussed in the next sections for bosons and fermions.

B.1.1 Bosonic fluctuations

The bosonic quadratic fluctuation Lagrangian is

$$\begin{aligned} \widetilde{\mathcal{L}}_B = & -\partial_a \tilde{t} \partial^a \tilde{t} - \mu_t^2 \tilde{t}^2 + \partial_a \tilde{\phi} \partial^a \tilde{\phi} + \mu_\phi^2 \tilde{\phi}^2 \\ & + 4 \tilde{\rho} (\kappa \sinh \rho \partial_0 \tilde{t} - w \cosh \rho \partial_0 \tilde{\phi}) + \partial_a \tilde{\rho} \partial^a \tilde{\rho} + \mu_\rho^2 \tilde{\rho}^2 \\ & + \partial_a \beta_u \partial^a \beta_u + \mu_\beta^2 \beta_u^2 + \partial_a \varphi \partial^a \varphi + \partial_a \chi_s \partial^a \chi_s + \mathcal{J}^2 \chi_s^2, \end{aligned} \quad (\text{B.1.2})$$

where

$$\begin{aligned} \mu_t^2 &= 2\rho'^2 - \kappa^2 + \mathcal{J}^2, & \mu_\phi^2 &= 2\rho'^2 - w^2 + \mathcal{J}^2, & \mu_\rho^2 &= 2\rho'^2 - w^2 - \kappa^2 + 2\mathcal{J}^2, \\ \mu_\beta^2 &= 2\rho'^2 + \mathcal{J}^2, & \mu_{\chi_s}^2 &= \mathcal{J}^2. \end{aligned} \quad (\text{B.1.3})$$

The two bosons β_i ($i = 1, 2$) are two AdS_5 fluctuations transverse to the AdS_3 subspace in which the string is moving, while φ, χ_s ($s = 1, 2, 3, 4$) are five fluctuations in S^5 .

We can write

$$\widetilde{\mathcal{L}}_B = (\tilde{t}, \tilde{\phi}, \tilde{\rho}) Q_B (\tilde{t}, \tilde{\phi}, \tilde{\rho})^T, \quad (\text{B.1.4})$$

where the Q_B operator is

$$Q_B = \begin{pmatrix} \partial^2 - \mu_t^2 & 0 & 2\kappa \sinh \rho \partial_0 \\ 0 & -\partial^2 + \mu_\phi^2 & -2w \cosh \rho \partial_0 \\ -2\kappa \sinh \rho \partial_0 & 2w \cosh \rho \partial_0 & -\partial^2 + \mu_\rho^2 \end{pmatrix}. \quad (\text{B.1.5})$$

B.1.2 Fermionic fluctuations

The fermionic lagrangian describes a system of 4+4 2d Majorana fermions. The result of [26] reads

$$\tilde{\mathcal{L}}_F = 2i \bar{\Psi} D_F \Psi, \quad D_F = \Gamma^a \partial_a + a(\sigma) \Gamma_{234} + b(\sigma) \Gamma_{129}, \quad (\text{B.1.6})$$

where

$$a(\sigma) = -\sqrt{\rho^2 + \mathcal{J}^2}, \quad b(\sigma) = \frac{\mathcal{J} \kappa w}{2(\rho^2 + \mathcal{J}^2)}. \quad (\text{B.1.7})$$

Taking the square

$$-D_F^2 = -\partial^a \partial_a + a^2 + b^2 + \Gamma_{1234} a' + \Gamma_{29}(b' + 2b \partial_\sigma) - 2ab \Gamma_{1349}. \quad (\text{B.1.8})$$

The matrices Γ_{1234} and Γ_{29} obey

$$\Gamma_{1234}^2 = 1, \quad \Gamma_{29}^2 = -1, \quad \Gamma_{1349}^2 = 1, \quad (\text{B.1.9})$$

$$\{\Gamma_{1234}, \Gamma_{29}\} = 0, \quad \{\Gamma_{1349}, \Gamma_{29}\} = 0, \quad \{\Gamma_{1234}, \Gamma_{1349}\} = 0, \quad (\text{B.1.10})$$

and can be replaced (up to the tensor product with a multiple of the identity) by

$$\Gamma_{1234} \equiv \sigma_3, \quad \Gamma_{29} \equiv i\sigma_1, \quad \Gamma_{1349} = -\sigma_2. \quad (\text{B.1.11})$$

In other words,

$$Q_F = -D_F^2 \equiv \begin{pmatrix} -\partial^2 + a^2 + b^2 + a' & i(2ab + b' + 2b \partial_\sigma) \\ i(-2ab + b' + 2b \partial_\sigma) & -\partial^2 + a^2 + b^2 - a' \end{pmatrix} \quad (\text{B.1.12})$$

B.2 The short string limit with $\mathcal{J} > 0$

We shall consider the short limit $\varepsilon \rightarrow 0$ with fixed \mathcal{J} . The solution of the equation of motion for $\rho(\sigma)$ is independent on \mathcal{J} and reads

$$\rho(\sigma) = \varepsilon \sin \sigma + \frac{\varepsilon^3}{12} \sin \sigma (\sin^2 \sigma - 3) + \frac{\varepsilon^5}{320} \sin \sigma (4 \sin^4 \sigma - 25 \sin^2 \sigma + 45) + \dots \quad (\text{B.2.1})$$

It is convenient to collect the explicit expansions of various terms appearing in the fluctuation lagrangians.

B.2.1 Bosonic terms

The expansion of the mixed terms in the operator Q_B is

$$\kappa \sinh \rho = \varepsilon \mathcal{J} \sin \sigma + \dots, \quad (\text{B.2.2})$$

$$w \cosh \rho = \sqrt{\mathcal{J}^2 + 1} + \frac{\varepsilon^2}{4} \frac{2(\mathcal{J}^2 + 1) \sin^2 \sigma + 1}{\sqrt{\mathcal{J}^2 + 1}} + \dots \quad (\text{B.2.3})$$

The expansion of the bosonic masses is

$$\mu_t^2 = \varepsilon^2 (2 \cos^2 \sigma - 1) + \dots, \quad (\text{B.2.4})$$

$$\mu_\phi^2 = -1 + \varepsilon^2 \left(2 \cos^2 \sigma - \frac{1}{2} \right) + \dots, \quad (\text{B.2.5})$$

$$\mu_\rho^2 = -1 + \varepsilon^2 \left(2 \cos^2 \sigma - \frac{3}{2} \right) + \dots, \quad (\text{B.2.6})$$

$$\mu_\beta^2 = \mathcal{J}^2 + 2\varepsilon^2 \cos^2 \sigma + \dots \quad (\text{B.2.7})$$

It is convenient to rotate the Q_B operator as

$$Q_B \rightarrow R_B^{-1} Q_B R_B, \quad R_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{i}{2} & \frac{i}{2} \end{pmatrix}. \quad (\text{B.2.8})$$

Then $Q_B = Q_B^{(0)} + \varepsilon Q_B^{(1)} + \dots$ and

$$Q_B^{(0)} = \begin{pmatrix} \omega^2 - n^2 & 0 & 0 \\ 0 & n^2 - \omega^2 - 2\sqrt{\mathcal{J}^2 + 1}\omega - 1 & 0 \\ 0 & 0 & n^2 - \omega^2 + 2\sqrt{\mathcal{J}^2 + 1}\omega - 1 \end{pmatrix}, \quad (\text{B.2.9})$$

when acting on functions $\sim e^{in\sigma}$. This is a good starting point for perturbation theory. The important remark is that now it is possible to send $\mathcal{J} \rightarrow 0$. We find the same result by sending first $\mathcal{J} \rightarrow 0$ and then $\varepsilon \rightarrow 0$. At finite $\mathcal{J} > 0$, the eigenvalues of $Q_B^{(0)}$ can be written as

$$\omega_n = \pm n, \quad (\text{B.2.10})$$

$$\omega_n = \pm(\sqrt{n^2 + \mathcal{J}^2} + \sqrt{1 + \mathcal{J}^2}), \quad (\text{B.2.11})$$

$$\omega_n = \pm(\sqrt{n^2 + \mathcal{J}^2} - \sqrt{1 + \mathcal{J}^2}). \quad (\text{B.2.12})$$

B.2.2 Fermionic terms

The expansion of the a and b functions appearing in the fermionic operator is

$$a(\sigma) = -\mathcal{J} - \frac{\varepsilon^2 \cos^2 \sigma}{2\mathcal{J}} + \dots, \quad (\text{B.2.13})$$

$$b(\sigma) = \frac{\sqrt{\mathcal{J}^2 + 1}}{2} + \varepsilon^2 \frac{\mathcal{J}^2 - 2(\mathcal{J}^2 + 1) \cos(2\sigma)}{8\mathcal{J}^2 \sqrt{\mathcal{J}^2 + 1}} + \dots \quad (\text{B.2.14})$$

Here we see a potential order of limits problem when $\varepsilon, \mathcal{J} \rightarrow 0$!

The explicit form of $Q_F^{(0)}$ is

$$Q_F^{(0)} = \begin{pmatrix} n^2 + \frac{5\mathcal{J}^2}{4} - \omega^2 + \frac{1}{4} & -(n + i\mathcal{J})\sqrt{\mathcal{J}^2 + 1} \\ i(in + \mathcal{J})\sqrt{\mathcal{J}^2 + 1} & n^2 + \frac{5\mathcal{J}^2}{4} - \omega^2 + \frac{1}{4} \end{pmatrix}. \quad (\text{B.2.15})$$

Again, it is convenient to rotate Q_F by a n -dependent rotation

$$Q_F \rightarrow R_F^{-1}(n) Q_F R_F(n), \quad R_F(n) = \begin{pmatrix} \sqrt{n^2 + \mathcal{J}^2} & -\sqrt{n^2 + \mathcal{J}^2} \\ n - i\mathcal{J} & n - i\mathcal{J} \end{pmatrix}. \quad (\text{B.2.16})$$

Then, acting on functions $\sim e^{in\sigma}$, the leading order is

$$Q_F^{(0)} = \begin{pmatrix} n^2 + \frac{5\mathcal{J}^2}{4} - \omega^2 - \sqrt{(\mathcal{J}^2 + 1)(n^2 + \mathcal{J}^2)} + \frac{1}{4} & 0 \\ 0 & n^2 - \omega^2 + \mathcal{J}^2 \left(\sqrt{\frac{n^2 + \mathcal{J}^2}{\mathcal{J}^2 + 1}} + \frac{5}{4} \right) + \sqrt{\frac{n^2 + \mathcal{J}^2}{\mathcal{J}^2 + 1}} + \frac{1}{4} \end{pmatrix} \quad (\text{B.2.17})$$

and its eigenfrequencies are

$$\omega_n = \pm \left(\sqrt{n^2 + \mathcal{J}^2} + \frac{1}{2} \sqrt{1 + \mathcal{J}^2} \right), \quad (\text{B.2.18})$$

$$\omega_n = \pm \left(\sqrt{n^2 + \mathcal{J}^2} - \frac{1}{2} \sqrt{1 + \mathcal{J}^2} \right). \quad (\text{B.2.19})$$

B.3 Balance of the ε^0 contributions

In the flat space limit, we have the following contributions from the various fields (in the standard notation)

multiplicity	field(s)	ω_n
1	(t, ϕ, ρ)	n $\frac{\sqrt{n^2 + \mathcal{J}^2} + \sqrt{\mathcal{J}^2 + 1}}{\sqrt{n^2 + \mathcal{J}^2} - \sqrt{\mathcal{J}^2 + 1}}$
2	β_u	$\sqrt{n^2 + \mathcal{J}^2}$
1	φ	n
4	χ_s	$\sqrt{n^2 + \mathcal{J}^2}$
4	Ψ	$\frac{\sqrt{n^2 + \mathcal{J}^2} + \frac{1}{2} \sqrt{\mathcal{J}^2 + 1}}{\sqrt{n^2 + \mathcal{J}^2} - \frac{1}{2} \sqrt{\mathcal{J}^2 + 1}}$
2	ghost	n

(B.3.1)

Summing with weight $(-1)^F$ we find cancellation of (a) massless contributions, (b) massive contributions $\sqrt{n^2 + \mathcal{J}^2}$, (c) constant n -independent terms.

B.4 Corrections to flat-space limit: Summary of results

We can compute the correction to the eigenfrequencies

$$\omega_n(\varepsilon) = \omega_n^{(0)} + \varepsilon \omega_n^{(1)} + \varepsilon^2 \omega_n^{(2)} + \dots \quad (\text{B.4.1})$$

by solving the (coupled) differential equations $\mathcal{D}(\partial_\sigma; \omega, n)\Phi_n = 0$ for the various field(s) Φ_n . Imposing periodic boundary conditions we determine ω , order by order at small ε . This procedure gives also the correction

$$\Phi_n = e^{in\sigma} + \varepsilon \Phi_n^{(1)} + \varepsilon^2 \Phi_n^{(2)} + \dots \quad (\text{B.4.2})$$

We find a zero correction $\omega^{(1)}$ in all cases. The second order correction is smooth for $|n| \neq 1$ otherwise some of the $\Phi_n^{(\ell)}$ coefficients can have singularities when $n \rightarrow \pm 1$.

For generic modes $|n| \neq 1$, the **summary list** of the second order corrections is:

$$\omega_n^{(t,\rho,\phi),I} = n, \quad (\text{B.4.3})$$

$$\omega_n^{(t,\rho,\phi),II} = \sqrt{n^2 + \mathcal{J}^2} + \sqrt{1 + \mathcal{J}^2} + \varepsilon^2 \frac{2\mathcal{J}^2 + 2 + \sqrt{(\mathcal{J}^2 + 1)(\mathcal{J}^2 + n^2)}}{4(\mathcal{J}^2 + 1)\sqrt{\mathcal{J}^2 + n^2}} + \dots, \quad (\text{B.4.4})$$

$$\omega_n^{(t,\rho,\phi),III} = \sqrt{n^2 + \mathcal{J}^2} - \sqrt{1 + \mathcal{J}^2} + \varepsilon^2 \frac{2\mathcal{J}^2 + 2 - \sqrt{(\mathcal{J}^2 + 1)(\mathcal{J}^2 + n^2)}}{4(\mathcal{J}^2 + 1)\sqrt{\mathcal{J}^2 + n^2}} + \dots, \quad (\text{B.4.5})$$

$$\omega_n^\beta = \sqrt{n^2 + \mathcal{J}^2} + \varepsilon^2 \frac{1}{2\sqrt{\mathcal{J}^2 + n^2}} + \dots, \quad (\text{B.4.6})$$

$$\omega_n^\varphi = n, \quad (\text{B.4.7})$$

$$\omega_n^\chi = \sqrt{\mathcal{J}^2 + n^2}, \quad (\text{B.4.8})$$

$$\omega_n^{\Psi,I} = \sqrt{n^2 + \mathcal{J}^2} + \frac{1}{2}\sqrt{1 + \mathcal{J}^2} + \varepsilon^2 \frac{2\mathcal{J}^2 + 2 + \sqrt{(\mathcal{J}^2 + 1)(\mathcal{J}^2 + n^2)}}{8(\mathcal{J}^2 + 1)\sqrt{\mathcal{J}^2 + n^2}} + \dots \quad (\text{B.4.9})$$

$$\omega_n^{\Psi,II} = \sqrt{n^2 + \mathcal{J}^2} - \frac{1}{2}\sqrt{1 + \mathcal{J}^2} + \varepsilon^2 \frac{2\mathcal{J}^2 + 2 - \sqrt{(\mathcal{J}^2 + 1)(\mathcal{J}^2 + n^2)}}{8(\mathcal{J}^2 + 1)\sqrt{\mathcal{J}^2 + n^2}} + \dots \quad (\text{B.4.10})$$

$$\omega_n^{\text{ghost}} = n. \quad (\text{B.4.11})$$

At the special values $n = \pm 1$ we find potential singularities in the corrections to $\Phi_{\pm 1}$ for various modes although the correction $\omega_{\pm 1}^{(2)}$ is smooth. In all cases, with the exception of the bosonic mode $(t, \rho, \phi)^{III}$, what happens is that the true frequencies associated with $n = \pm 1$ obey

$$\omega_{n=1}^{(2),\text{true}} + \omega_{n=-1}^{(2),\text{true}} = 2\omega_{n=1}^{(2), \text{from the generic-}n \text{ summary list}}. \quad (\text{B.4.12})$$

This means that we can safely use the summary list expressions if we are going to sum over all frequencies as is the case for the computation of the one-loop energy.

The only non trivial modification concerns $\omega_n^{(t,\rho,\phi),III}$ at $n = \pm 1$ that must be replaced by zero. Indeed, for $|n| = 1$ there are two independent periodic solutions that have precisely $\omega^{(2)} \equiv 0$ as discussed in the following sections.

For $|n| \neq 1$ we have

$$\omega_n^{(t,\rho,\phi),I} + \omega_n^{(t,\rho,\phi),II} + \omega_n^{(t,\rho,\phi),III} + 2\omega_n^\beta + \omega_n^\varphi + 4\omega_n^\chi - 4\omega_n^{\Psi,I} - 4\omega_n^{\Psi,II} - 2\omega_n^{\text{ghost}} = 0, \quad (\text{B.4.13})$$

where the frequencies are those reported in the summary list. Taking into account that $\kappa = \mathcal{J} + \dots$ and that $\varepsilon^2 = \frac{2\mathcal{S}}{\sqrt{\mathcal{J}^2 + 1}} + \dots$, we find that the $|n| = 1$ modes give

$$E_1 = \frac{1}{2\kappa} \cdot 2 \left(\omega_1^{(t,\rho,\phi),I} + \omega_1^{(t,\rho,\phi),II} + 2\omega_1^\beta + \omega_1^\varphi + 4\omega_1^\chi - 4\omega_1^{\Psi,I} - 4\omega_1^{\Psi,II} - 2\omega_1^{\text{ghost}} \right) =$$

$$= -\frac{1}{\kappa} \omega_1^{(t,\rho,\phi),III} = -\frac{\mathcal{S}}{2(\mathcal{J} + \mathcal{J}^3)} + \mathcal{O}(\mathcal{S}^2), \quad (\text{B.4.14})$$

in complete agreement with [33].

We conclude this long appendix by a detailed example of the frequency computation, including special low modes.

B.5 A detailed example of calculation: The β mode

B.5.1 Generic n

This is the simplest case. One has to solve the equation

$$\left(\frac{d^2}{d\sigma^2} + \omega^2 - \mu_\beta^2 \right) \Phi_n(\sigma) = 0, \quad (\text{B.5.1})$$

where boundary conditions are periodic and the following perturbative Ansatz is imposed

$$\omega = \sqrt{n^2 + \mathcal{J}^2} + \varepsilon^2 \omega_n^{(2)} + \dots, \quad (\text{B.5.2})$$

$$\Phi_n(\sigma) = e^{i n \sigma} + \varepsilon^2 (z_1 e^{i(n+2)\sigma} + z_2 e^{i(n-2)\sigma}) + \dots. \quad (\text{B.5.3})$$

Solving the equation at the first non-trivial order determines for generic n

$$z_1 = -\frac{1}{8(n+1)}, \quad z_2 = \frac{1}{8(n-1)}, \quad \omega_n^{(2)} = \frac{1}{2\sqrt{n^2 + \mathcal{J}^2}}. \quad (\text{B.5.4})$$

B.5.2 Special values $n = \pm 1$

One has to consider the special values $n = \pm 1$ at which the above solution breaks down. The problem is that for $n = \pm 1$ the solutions starting as $e^{\pm i \sigma}$ are mixed up. Thus, the correct Ansatz in this case is

$$\omega = \sqrt{1 + \mathcal{J}^2} + \varepsilon^2 \omega^{(2)} + \dots, \quad (\text{B.5.5})$$

$$\Phi(\sigma) = e^{i \sigma} + \alpha e^{-i \sigma} + \varepsilon^2 (z_1 e^{3i \sigma} + z_2 e^{-3i \sigma}) + \dots, \quad (\text{B.5.6})$$

and also the mixing parameter α has to be determined. Plugging into the differential equation one finds two solutions

$$\alpha = +1, \quad z_1 = +z_2 = -\frac{1}{16}, \quad \omega^{(2)} = \frac{3}{4\sqrt{1 + \mathcal{J}^2}}, \quad (\text{B.5.7})$$

$$\alpha = -1, \quad z_1 = -z_2 = -\frac{1}{16}, \quad \omega^{(2)} = \frac{1}{4\sqrt{1 + \mathcal{J}^2}}. \quad (\text{B.5.8})$$

The sum of the two different values of $\omega^{(2)}$ is twice the naive value which is obtained by evaluating the generic result $\omega_n^{(2)}$ at $n = 1$. Hence, if we sum over all frequencies then we can simply use the expression $\omega_n^{(2)}$.

Of course, in this simple case, the values $\alpha = \pm 1$ tell us that parity would have been enough to disentangle the two frequencies. Nevertheless, the above procedure is general.

C Separating out wrapping terms in infinite sums

We often have to compute complicated sums depending on \mathcal{J} and we want to separate out the exponentially suppressed contribution at large \mathcal{J} . We illustrate how to practically treat sums of the kind occurring in the computation by discussing the nice example of

$$S(\mathcal{J}, \alpha) = \sum_{n=-\infty}^{\infty} f(n), \quad f(n) = \frac{1}{(n^2 - \alpha^2)^2} \frac{1}{\sqrt{n^2 + \mathcal{J}^2}}, \quad 0 < \alpha < 1, \mathcal{J} > 0. \quad (\text{C.1})$$

We would like to apply Euler-McLaurin formula since, up to a remainder that we don't write explicitly,

$$\begin{aligned} \frac{1}{2}f(m) + f(m+1) + \dots + f(n-1) + \frac{1}{2}f(n) &= \int_m^n f(x)dx \\ &+ \sum_{k=2}^{\infty} \frac{B_k}{k!} \left[f^{(k-1)}(n) - f^{(k-1)}(m) \right] + \text{remainder}. \end{aligned} \quad (\text{C.2})$$

When $m \rightarrow -\infty$ and $n \rightarrow +\infty$, the derivative terms vanish and only the integral remains. Actually, this means that there is a remainder in the Euler formula is generically nonzero, but is exponentially suppressed at large \mathcal{J} . However, the integral diverges at $n^2 = \alpha^2$ and the singularity is not integrable.

The trick is then to write

$$f(n) = \frac{1}{(n^2 - \alpha^2)} g(n^2), \quad g(n^2) = \frac{1}{\sqrt{n^2 + \mathcal{J}^2}}, \quad (\text{C.3})$$

and to use

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{g(n^2)}{(n^2 - \alpha^2)^2} &= \sum_{n=-\infty}^{\infty} \frac{g(n^2) - g(\alpha^2) - (n^2 - \alpha^2) g'(\alpha^2)}{(n^2 - \alpha^2)^2} \\ &+ g(\alpha^2) \sum_{n=-\infty}^{\infty} \frac{1}{(n^2 - \alpha^2)^2} + g'(\alpha^2) \sum_{n=-\infty}^{\infty} \frac{1}{n^2 - \alpha^2}. \end{aligned} \quad (\text{C.4})$$

The last two terms can be summed exactly and the first sum can be converted into an integral up to the remainder. The result is then

$$\begin{aligned} S(\mathcal{J}, \alpha) &= \frac{\pi (\cot(\pi\alpha) (2\alpha^2 + \mathcal{J}^2) + \pi\alpha \csc^2(\pi\alpha) (\alpha^2 + \mathcal{J}^2))}{2\alpha^3 (\alpha^2 + \mathcal{J}^2)^{3/2}} \\ &+ \frac{(2\alpha^2 + \mathcal{J}^2) \left(\log\left(\frac{\sqrt{\alpha^2 + \mathcal{J}^2}}{\alpha} + 1\right) - \log\left(\frac{\sqrt{\alpha^2 + \mathcal{J}^2}}{\alpha} - 1\right) \right)}{2(\alpha^2 + \mathcal{J}^2)^{3/2}} - \frac{\alpha}{\alpha^2 + \mathcal{J}^2} \\ &+ \frac{\dots}{\alpha^3} + \text{exponentially suppressed}. \end{aligned} \quad (\text{C.5})$$

D Details of AC shifts of frequencies

The shifts appearing in the calculation of frequencies according to the quantization of the algebraic curve are

$$\Delta\Omega_S = -\mathcal{J}, \quad (\text{D.1})$$

$$\begin{aligned} \Delta\Omega_1 = & \left(-\sqrt{\mathcal{J}^2+1}-\mathcal{J}\right) + \left(-\frac{1}{2\mathcal{J}^2+2}-\frac{1}{\mathcal{J}\sqrt{\mathcal{J}^2+1}}\right)\mathcal{S} \\ & + \frac{(3\mathcal{J}^5+9\mathcal{J}^3+28\sqrt{\mathcal{J}^2+1}\mathcal{J}^2+8\sqrt{\mathcal{J}^2+1}+12\sqrt{\mathcal{J}^2+1}\mathcal{J}^4)\mathcal{S}^2}{16\mathcal{J}^3(\mathcal{J}^2+1)^{5/2}} + O(\mathcal{S}^3), \end{aligned} \quad (\text{D.2})$$

$$\begin{aligned} \Delta\Omega_A = & \left(\sqrt{\mathcal{J}^2+1}-\mathcal{J}\right) + \left(\frac{1}{2\mathcal{J}^2+2}-\frac{1}{\mathcal{J}\sqrt{\mathcal{J}^2+1}}\right)\mathcal{S} \\ & + \frac{(-3\mathcal{J}^5-9\mathcal{J}^3+28\sqrt{\mathcal{J}^2+1}\mathcal{J}^2+8\sqrt{\mathcal{J}^2+1}+12\sqrt{\mathcal{J}^2+1}\mathcal{J}^4)\mathcal{S}^2}{16\mathcal{J}^3(\mathcal{J}^2+1)^{5/2}} + O(\mathcal{S}^3), \end{aligned} \quad (\text{D.3})$$

$$\begin{aligned} \Delta\Omega_3 = & \left(\frac{\sqrt{\mathcal{J}^2+1}}{2}-\mathcal{J}\right) + \left(\frac{1}{4(\mathcal{J}^2+1)}-\frac{1}{2\mathcal{J}\sqrt{\mathcal{J}^2+1}}\right)\mathcal{S} \\ & + \frac{(-3\mathcal{J}^5-9\mathcal{J}^3+28\sqrt{\mathcal{J}^2+1}\mathcal{J}^2+8\sqrt{\mathcal{J}^2+1}+12\sqrt{\mathcal{J}^2+1}\mathcal{J}^4)\mathcal{S}^2}{32\mathcal{J}^3(\mathcal{J}^2+1)^{5/2}} + O(\mathcal{S}^3), \end{aligned} \quad (\text{D.4})$$

$$\begin{aligned} \Delta\Omega_4 = & \left(-\frac{\sqrt{\mathcal{J}^2+1}}{2}-\mathcal{J}\right) + \left(-\frac{1}{2\mathcal{J}\sqrt{\mathcal{J}^2+1}}-\frac{1}{4(\mathcal{J}^2+1)}\right)\mathcal{S} \\ & + \frac{(3\mathcal{J}^5+9\mathcal{J}^3+28\sqrt{\mathcal{J}^2+1}\mathcal{J}^2+8\sqrt{\mathcal{J}^2+1}+12\sqrt{\mathcal{J}^2+1}\mathcal{J}^4)\mathcal{S}^2}{32\mathcal{J}^3(\mathcal{J}^2+1)^{5/2}} + O(\mathcal{S}^3). \end{aligned} \quad (\text{D.5})$$

E Details of AC-WS regularization matching

The quantities $\Delta_A^{(0)}$ and $\Omega_A^{(0)}$ defined in Section (3.4) for the various polarizations $A = S, A, 1, 3, 4$ have the explicit values:

$$\Delta_S^{(0)} = 0, \quad (\text{E.1})$$

$$\Delta_A^{(0)} = -\frac{\mathcal{S}}{\mathcal{J}^2} + \frac{\sqrt{\mathcal{J}^2+1}\mathcal{S}^2}{2\mathcal{J}^4} + O(\mathcal{S}^3), \quad (\text{E.2})$$

$$\Delta_1^{(0)} = \frac{\mathcal{S}}{\mathcal{J}^2} - \frac{\sqrt{\mathcal{J}^2+1}\mathcal{S}^2}{2\mathcal{J}^4} + O(\mathcal{S}^3), \quad (\text{E.3})$$

$$\Delta_3^{(0)} = -\frac{\mathcal{S}}{2\mathcal{J}^2} + \frac{\sqrt{\mathcal{J}^2+1}\mathcal{S}^2}{4\mathcal{J}^4} + O(\mathcal{S}^3), \quad (\text{E.4})$$

$$\Delta_4^{(0)} = \frac{\mathcal{S}}{2\mathcal{J}^2} - \frac{\sqrt{\mathcal{J}^2+1}\mathcal{S}^2}{4\mathcal{J}^4} + O(\mathcal{S}^3), \quad (\text{E.5})$$

and

$$\Omega_S^{(0)} = -\mathcal{J} + O(\mathcal{S}^3), \quad (\text{E.6})$$

$$\Omega_A^{(0)} = -\mathcal{J} - \frac{\left(\frac{\mathcal{J}}{\sqrt{\mathcal{J}^2+1}}+1\right)\mathcal{S}}{\mathcal{J}^2} \quad (\text{E.7})$$

$$\Omega_1^{(0)} = -\mathcal{J} + \frac{\left(1 - \frac{\mathcal{J}}{\sqrt{\mathcal{J}^2+1}}\right) \mathcal{S}}{\mathcal{J}^2} + \frac{(3\mathcal{J}^6 + 8\mathcal{J}^4 + 7\mathcal{J}^2 + 2\sqrt{\mathcal{J}^2+1}\mathcal{J} + 3\sqrt{\mathcal{J}^2+1}\mathcal{J}^5 + 7\sqrt{\mathcal{J}^2+1}\mathcal{J}^3 + 2) \mathcal{S}^2}{4\mathcal{J}^4 (\mathcal{J}^2 + 1)^{5/2}} + O(\mathcal{S}^3), \quad (\text{E.8})$$

$$+ \frac{(-3\mathcal{J}^6 - 8\mathcal{J}^4 - 7\mathcal{J}^2 + 2\sqrt{\mathcal{J}^2+1}\mathcal{J} + 3\sqrt{\mathcal{J}^2+1}\mathcal{J}^5 + 7\sqrt{\mathcal{J}^2+1}\mathcal{J}^3 - 2) \mathcal{S}^2}{4\mathcal{J}^4 (\mathcal{J}^2 + 1)^{5/2}} + O(\mathcal{S}^3),$$

$$\Omega_3^{(0)} = -\mathcal{J} - \frac{\left(\frac{\mathcal{J}}{\sqrt{\mathcal{J}^2+1}} + 1\right) \mathcal{S}}{2\mathcal{J}^2} + \frac{(3\mathcal{J}^6 + 8\mathcal{J}^4 + 7\mathcal{J}^2 + 2\sqrt{\mathcal{J}^2+1}\mathcal{J} + 3\sqrt{\mathcal{J}^2+1}\mathcal{J}^5 + 7\sqrt{\mathcal{J}^2+1}\mathcal{J}^3 + 2) \mathcal{S}^2}{8\mathcal{J}^4 (\mathcal{J}^2 + 1)^{5/2}} + O(\mathcal{S}^3), \quad (\text{E.9})$$

$$+ \frac{(-3\mathcal{J}^6 - 8\mathcal{J}^4 - 7\mathcal{J}^2 + 2\sqrt{\mathcal{J}^2+1}\mathcal{J} + 3\sqrt{\mathcal{J}^2+1}\mathcal{J}^5 + 7\sqrt{\mathcal{J}^2+1}\mathcal{J}^3 - 2) \mathcal{S}^2}{8\mathcal{J}^4 (\mathcal{J}^2 + 1)^{5/2}} + O(\mathcal{S}^3),$$

$$\Omega_4^{(0)} = -\mathcal{J} + \frac{\left(1 - \frac{\mathcal{J}}{\sqrt{\mathcal{J}^2+1}}\right) \mathcal{S}}{2\mathcal{J}^2} + \frac{(-3\mathcal{J}^6 - 8\mathcal{J}^4 - 7\mathcal{J}^2 + 2\sqrt{\mathcal{J}^2+1}\mathcal{J} + 3\sqrt{\mathcal{J}^2+1}\mathcal{J}^5 + 7\sqrt{\mathcal{J}^2+1}\mathcal{J}^3 - 2) \mathcal{S}^2}{8\mathcal{J}^4 (\mathcal{J}^2 + 1)^{5/2}} + O(\mathcal{S}^3). \quad (\text{E.10})$$

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